COMBINATORIAL RIGIDITY FOR SOME INFINITELY RENORMALIZABLE UNICRITICAL POLYNOMIALS

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ABSTRACT. Here we prove that *infinitely renormalizable* unicritical polynomials $P_c : z \mapsto z^d + c$, with $c \in \mathbb{C}$, satisfying *a priori bounds* and a certain "combinatorial" condition are *combinatorially rigid*. This implies the local connectivity of the connectedness loci (the Mandelbrot set when $d = 2$) at the corresponding parameters.

1. INTRODUCTION

The *Multibrot set* $\mathcal{M}_d$, or the *connectedness locus* of the unicritical polynomials, is the set of parameter values c in $\mathbb{C}$ for which the *Julia set* of $P_c : z \mapsto z^d + c$ is connected. The set $\mathcal{M}_2$ is the well-known *Mandelbrot* set.

There is a way of defining graded partitions (puzzle pieces) of the Multibrot set such that the dynamics of the maps P_c in each piece have some special combinatorial property. All maps in a given piece of a partition of a certain level are called *combinatorially equivalent* up to that level. Conjecturally, combinatorially equivalent (up to all levels) *non-hyperbolic* maps in this family are *conformally conjugate*. As stated in [DH85a] for $d = 2$, this *rigidity conjecture* is equivalent to the local connectivity of the Mandelbrot set, and it naturally extends to degree d unicritical polynomials. In the quadratic case, this conjecture is formulated as MLC by Douady and Hubbard. They also proved there that MLC implies the density of *hyperbolic polynomials* in the space of quadratic polynomials. These discussions have been extended to degree d unicritical polynomials by Schleicher in [Sch04].

In the 1990's, Yoccoz proved that $\mathcal{M}_2$ is locally connected at all non-hyperbolic parameter values which are at most finitely *renormalizable*. He also proved the local connectivity of the Julia sets of these maps with all periodic points repelling (see [Hub93]). The degree two assumption was essential in his proof.

In [Lyu97], Lyubich proved the combinatorial rigidity conjecture for a class of infinitely renormalizable quadratic polynomials. These are quadratic polynomials satisfying a *secondary limbs condition*, denoted by $\mathcal{SL}$, with sufficiently high return times. The proof in this case also relies on the degree two assumption.

The local connectivity of the Julia sets of degree d unicritical polynomials which are at most finitely renormalizable and with all periodic points repelling has been shown in [KL09a]. Their proof is based on "controlling" the geometry of a *modified principal nest*. The same controlling technique has been used to settle the rigidity problem for these parameters in [AKLS09].

Recently, the *a priori* bounds property, a type of compactness on renormalization levels, has been established for more parameters. In [Kah06], it is proved for infinitely *primitively* renormalizable maps of bounded type. In [KL08], it is proved

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for parameters satisfying a *decorations* condition and in [KL09b], under a *molecule* condition. Here we prove that the *a priori* bounds property, under the $\mathcal{SL}$ condition, implies the combinatorial rigidity conjecture for infinitely renormalizable maps. The $\mathcal{SL}$ class includes all parameters for which *a priori* bounds is known to us.

Theorem (Rigidity)**.** *Let P_c be an infinitely renormalizable degree d unicritical polynomial satisfying the a priori bounds and $\mathcal{SL}$ conditions. Then P_c is combinatorially rigid.*

This result was proved in part II of [Lyu97] for quadratic polynomials. That proof as well as the one presented here are based on the Sullivan-Thurston pullback argument. However, the one in [Lyu97] uses linear growth of certain moduli along the principal nest which does not hold for arbitrary degree unicritical polynomials. It turns out that the definite modulus of certain annuli in a modified principal nest introduced in [AKLS09] helps us to easily "pass" over the principal nest. This makes the whole construction simpler and more general so that we can include unicritical polynomials of arbitrary degree. Combining the above theorem with [KL08] and [KL09b] we obtain the following:

Corollary. *Assume that P and $\tilde{P}$ are combinatorially equivalent infinitely renormalizable unicritical polynomials with one of the following conditions:*

- *P and $\tilde{P}$ are quadratic and satisfy the molecule condition,*

  or,
- *P and $\tilde{P}$ have arbitrary degree and satisfy the decoration condition.*

Then, P and $\tilde{P}$ are conformally equivalent.

The rigidity problem for a separate combinatorial class of quadratics is treated by a wholly different approach in [Lev09] which does not involve the *a priori* bounds property.

The rigidity conjecture for real polynomials has been fully established over the last twenty years. The quadratic case was accomplished, independently, in [Lyu97] and [GŚ98]. The real multi-critical case was treated in [LvS98]. One may refer to these for further references. Our result can be applied to real unicritical polynomials as well. Therefore, combining with [Sul92], it gives a new proof of the density of hyperbolicity in the family $x \to x^{2k} + c$, $k = 1, 2, \ldots$, which was proved earlier in [KSvS07].

The structure of the paper is as follows. In §2 we introduce the basics of holomorphic dynamics required for our work. In §3, Yoccoz puzzle pieces are defined, the modified principal nest is introduced, and combinatorics of unicritical polynomials is discussed. The proof of the main theorem, presented in Section 4, is reduced to the existence of a *Thurston conjugacy* by the pullback method. To build such a conjugacy, we start with a topological conjugacy on the whole complex plane and then step by step, on finer and finer scales, replace this homeomorphism by quasi-conformal maps while sacrificing the equivariance property but staying in the "right" homotopy class. At the end one obtains a quasi conformal map on the complex plane homotopic to a topological conjugacy relative the post-critical set, that is, a Thurston conjugacy.

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2. Polynomials and the connectedness loci

2.1. **External rays and Equipotentials.** One can read more about the following basics of holomorphic dynamics in [Mil06] and [Bra94].

Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree d: $f(z) = z^d + a_1 z^{d-1} + \cdots + a_d$. Infinity is a super attracting fixed point of f whose *basin of attraction* is defined as

$$D_f(\infty) := \{z \in \mathbb{C} : f^n(z) := \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ times}}(z) \to \infty \text{ as } n \to \infty\}.$$

The complement of $D_f(\infty)$ is called the *filled Julia* set: $K(f) := \mathbb{C} \setminus D_f(\infty)$. The *Julia set*, $J(f)$, is defined as the boundary of $K(f)$. It is well-known that the Julia set and the filled Julia set of a polynomial are connected if and only if the orbit of all critical points stay bounded under iteration.

With f as above, there exists a conformal change of coordinate B_f, the *Böttcher coordinate*, which conjugates f to the d-th power map $z \mapsto z^d$ throughout some neighborhood of infinity U_f. That is,

$$B_f : U_f \to \{z \in \mathbb{C} : |z| > r_f \geq 1\} \tag{2.1}$$

with $B_f(f(z)) = (B_f(z))^d$, and $B_f(z) \sim z$ as $z \to \infty$.

In particular, if the filled Julia set is connected, B_f coincides with the Riemann mapping of $D_f(\infty)$ onto the complement of the closed unit disk normalized to be tangent to the identity map at infinity.

The *external ray* (or *ray* for short) *of angle* θ is defined as

$$R^\theta = R_f^\theta := B_f^{-1}\{re^{i\theta} : r_f < r < \infty\}.$$

The *equipotential of level* $r > r_f$ is defined as

$$E^r = E_f^r := B_f^{-1}\{re^{i\theta} : 0 \leq \theta \leq 2\pi\}.$$

The *equivariance* property of the map B_f, i.e. $B_f(f(z)) = (B_f(z))^d$, implies that $f(R^\theta) = R^{d\theta}$, and $f(E^r) = E^{r^d}$.

A ray R^θ is called *periodic* of period p if $f^p(R^\theta) = R^\theta$. A ray is fixed (has period 1) if and only if θ is a rational number of the form $2\pi j/(d-1)$. By definition, a ray R^θ *lands* at a well defined point z in $J(f)$, if the limiting value of the ray R^θ (as $r \to 1$) exists and is equal to z. Such a point z in $J(f)$ is called the *landing point* of the ray R^θ. The following theorem characterizes the landing points of the periodic rays. See [DH85a] for further discussions.

Theorem 2.1. *Let f be a polynomial of degree $d \geq 2$ with connected Julia set. Every periodic ray lands at a well defined periodic point which is either repelling or parabolic. Vice versa, every repelling or parabolic periodic point is the landing point of at least one, and at most finitely many periodic rays with the same ray period.*

In particular, this theorem implies that the external rays landing at a periodic point are organized in several cycles. Suppose $\overline{a} = \{a_k\}_{k=0}^{p-1}$ is a repelling or parabolic cycle of f. Let $\mathfrak{R}(a_k)$ denote the union of all the external rays landing at a_k.

The configuration

$$\mathfrak{R}(\overline{a}) = \bigcup_{k=0}^{p-1} \mathfrak{R}(a_k),$$

with the rays labeled by their external angles, is called the *periodic point portrait* of f associated to the cycle $\overline{a}$.

2.2. **Unicritical family and the connectedness locus.** Any degree d polynomial with only one critical point is conformally conjugate to some $P_c(z) = z^d + c$, with $c \in \mathbb{C}$. A case of special interest is the following fixed point portrait. The $d-1$ fixed rays $R^{2\pi j/(d-1)}$ land at $d-1$ (distinct) fixed points called β_j. Moreover, these are the only rays that land at β_j's. Therefore, these fixed points are *non-dividing*, that is, $K(P_c) \setminus \beta_j$ is connected for any j. If the other fixed point called α is also repelling, there are at least two rays that land at it. Thus, the α-fixed point is *dividing* and by Theorem 2.1, the rays landing at α are permuted under the dynamics. The following statement has been shown in [Mil00b] for quadratic polynomials. The same ideas apply to prove it for degree d unicritical polynomials.

Proposition 2.2. *If at least two rays land at the α fixed point of P_c, we have:*

- *the component of $\mathbb{C} \setminus P_c^{-1}(\mathfrak{R}(\alpha))$ containing the critical value is a sector bounded by two external rays;*
- *the component of $\mathbb{C} \setminus P_c^{-1}(\mathfrak{R}(\alpha))$ containing the critical point is a region bounded by $2d$ external rays landing in pairs at the points $e^{2\pi j/d}\alpha$, for $j = 0, 1, \ldots, d-1$.*

The *connectedness locus* $\mathcal{M}_d$, or the *Multibrot* set of degree d, is defined as the set of parameters c in $\mathbb{C}$ for which $J(P_c)$ is connected. In particular, $\mathcal{M}_2$ is the well-known *Mandelbrot* set. See Figures 1 and 2.

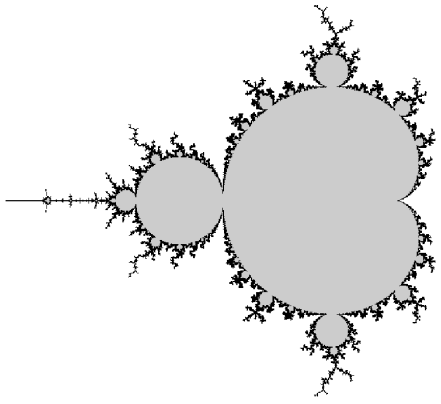

FIGURE 1. The Mandelbrot set. The gray regions show the interior of $\mathcal{M}_2$ and darker points show its boundary.

A well-known result due to Douady and Hubbard [DH85a] shows that these connectedness loci are connected. Their argument is based on considering the explicit conformal isomorphism

$$\mathcal{B}_d : \mathbb{C} \setminus \mathcal{M}_d \to \{z \in \mathbb{C} : |z| > 1\}$$

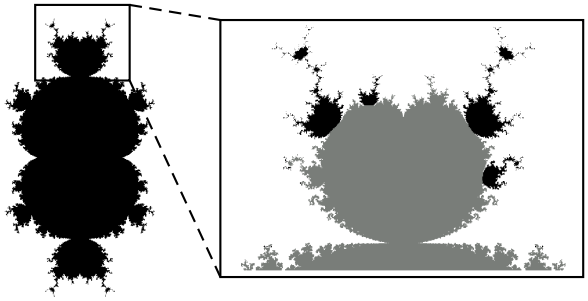

FIGURE 2. Figure on the left shows the connectedness locus $\mathcal{M}_3$. The figure on the right is an enlargement of a primary limb in $\mathcal{M}_3$. The dark regions in the right box show some of the secondary limbs.

given by $\mathcal{B}_d(c) := B_c(c)$, where B_c is the Böttcher coordinate (2.1) of P_c.

By means of the conformal isomorphism $\mathcal{B}_d$, *parameter external rays* R_θ and *equipotentials* E_r are defined as $\mathcal{B}_d$-preimages of straight rays going to infinity and round circles around 0, respectively.

A polynomial P_c (also the corresponding parameter c) is called *hyperbolic* if P_c has an attracting periodic point. This attracting periodic point necessarily attracts orbit of the finite critical point. The set of hyperbolic parameters in $\mathcal{M}_d$, which is open by definition, is a union of some components of $\operatorname{int}\mathcal{M}_d$. These components are called the *hyperbolic components*.

The main hyperbolic component is defined as the set of parameter values c for which P_c has an attracting fixed point. Outside of the closure of this set all fixed points become repelling. Consider a parameter c in a hyperbolic component $\mathscr{H} \subset \operatorname{int}\mathcal{M}_d$, and suppose that $\overline{b}_c$ denotes the corresponding attracting cycle with period $k > 1$. On the boundary of $\mathscr{H}$ this cycle becomes neutral, and there are $d-1$ parameters $c_i \in \partial\mathscr{H}$ where P_{c_i} has a *parabolic* cycle with multiplier equal to one. One of these parameters, which is called the *root* of $\mathscr{H}$ and is denoted by c_{root}, divides the connectedness locus into two pieces. Indeed, any hyperbolic component has one root and $d-2$ *co-roots*. The root is the landing point of two parameter rays, while every co-root is the landing point of a single parameter ray. See Figure 3. For a proof of these statements one may consult [DH85a], for the quadratic polynomials, and [Sch04] for arbitrary degree unicritical polynomials.

If c belongs to a hyperbolic component $\mathscr{H}$ which is not the main hyperbolic component of the connectedness locus, the *basin of attraction* of its attracting cycle $\bar{b}_c$, denoted by A_c, is defined as the set of points $z \in \mathbb{C}$ with $\langle P_c^n(z)\rangle_{n=0}^{\infty}$ converges to the cycle $\overline{b}_c$. The boundary of the component of A_c containing c is a Jordan curve which we denote it by D_c. The map P_c^k on D_c is topologically conjugate to $\theta \mapsto d\theta$ on the unit circle. Therefore, there are $d-1$ fixed points of P_c^k on this Jordan curve which are repelling periodic points of P_c of period dividing the period of $\bar{b}_c$ (its period can be strictly less than the period of $\bar{b}_c$). Among all rays landing at these repelling periodic points, let θ_1 and θ_2 be the angles of the external rays bounding the sector containing the critical value of P_c (See Figure 3). The following theorem makes a connection between external rays R^{θ_1}, R^{θ_2} and the parameter external rays R_{θ_1}, R_{θ_2}. See [DH85a] and [Sch04] for further details.

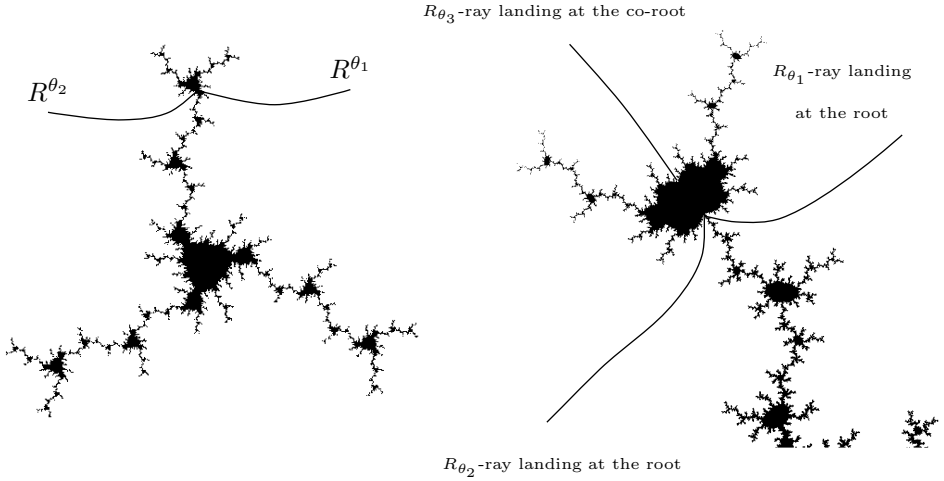

FIGURE 3. The figure on the left shows a primitively renormalizable Julia set, as well as the external rays R^{θ_1} and R^{θ_2} landing at the corresponding repelling periodic point. The figure on the right is the corresponding primitive little Multibrot copy. It also shows the parameter external rays R_{θ_1} and R_{θ_2} landing at the root point.

Theorem 2.3. *The parameter external rays R_{θ_1} and R_{θ_2} land at the root of $\mathscr{H}$. Moreover, these are the only rays that land at this point.*

The closure of R_{θ_1} and R_{θ_2} cut the parameter plane into two components. The one containing $\mathscr{H}$ with the root point attached to it is called the *wake* $W_{\mathscr{H}}$. So a wake is an open set with a point attached to its boundary. Given a wake $W_{\mathscr{H}}$ and an equipotential E_η, the *truncated wake* $W_{\mathscr{H}}(\eta)$ is the bounded component of $W_{\mathscr{H}} \backslash E_\eta$. The part of the connectedness locus contained in $W_{\mathscr{H}}$ is called the *limb* $\mathscr{L}_{\mathscr{H}}$ of the connectedness locus *originating* at $\mathscr{H}$. In other words, $\mathscr{L}_{\mathscr{H}} = W_{\mathscr{H}} \cap \mathcal{M}_d$. By definition, every limb is a closed set.

The wakes attached to the main hyperbolic component of $\mathcal{M}_d$ are called *primary wakes*. A limb associated to such a primary wake is called a *primary limb*. If $\mathscr{H}$ is a hyperbolic component attached to the main hyperbolic component, all the wakes attached to $\mathscr{H}$ (except $W_{\mathscr{H}}$ itself) are called *secondary wakes*. Similarly, the limb associated to a secondary wake is called a *secondary limb*. A *truncated limb* is obtained from a limb by removing a neighborhood of its root. Some secondary limbs are shown in Figure 2.

Given a parameter c in a hyperbolic component $\mathscr{H}$, we have the attracting cycle $\overline{b}_c$ as above, and the associated repelling cycle $\overline{a}_c$ that is the landing point of the external rays R^{θ_1} and R^{θ_2}. The following result gives the dynamical meaning of the parameter values in the wake $W_{\mathscr{H}}$ bounded by parameter external rays R_{θ_1} and R_{θ_2} (See [Sch04] for further details).

Theorem 2.4. *For parameters c in $W_{\mathscr{H}}\backslash\{root\}$, the cycle $\overline{a}_c$ stays repelling and, moreover, the isotopy type of the ray portrait $\mathfrak{R}(\overline{a}_c)$ is fixed throughout $W_{\mathscr{H}}$.*

2.3. **Polynomial-like mappings.** A polynomial-like map is a holomorphic proper branched covering of degree d, $f : U' \to U$, where U and U' are simply connected domains with U' compactly contained in U. For example, every polynomial can be viewed as a polynomial-like mapping once restricted to an appropriate neighborhood of its filled Julia set. This notion was introduced in [DH85b] to explain the presence of homeomorphic copies of the Mandelbrot set within the Mandelbrot set.

The filled Julia set $K(f)$ of a polynomial-like mapping $f : U' \to U$ is naturally defined as

$$K(f) := \{z \in \mathbb{C} : f^n(z) \in U', \text{ for } n = 0, 1, 2, \ldots\}.$$

The Julia set $J(f)$ is defined as the boundary of $K(f)$. These sets are connected if and only if $K(f)$ contains all critical points of f.

Two polynomial-like mappings f and g with Julia sets $J(f)$ and $J(g)$, respectively, are called *topologically conjugate* if there are choices of domains U, U', V, and V' as well as a homeomorphism $h : U \to V$ such that $f : U' \to U$ and $g : V' \to V$ are polynomial-like, with the same Julia sets $J(f)$ and $J(g)$, and $h \circ f = g \circ h$ on U'. They are called *quasi-conformally* (*conformally*, or *affinely*) conjugate if h can be chosen quasi-conformal (conformal, or affine, respectively). The notation $\mathrm{Dil}\,(h)$ is used for the quasi-conformal dilatation of a given quasi-conformal mapping h.

Two polynomial-like mappings f and g are *hybrid or internally equivalent* if there exists a quasi-conformal conjugacy h (q.c. conjugacy for short) between f and g such that $\overline{\partial}h = 0$ on $K(f)$. The following remarkable rigidity type theorem due to Douady and Hubbard [DH85b] states that the dynamics of a polynomial-like mapping is essentially the same as the one of a polynomial.

Theorem 2.5 (Straightening)**.** *Every polynomial-like mapping f is hybrid equivalent to (a suitable restriction of) a polynomial P of the same degree. Moreover, P is unique up to affine conjugacy when $K(f)$ is connected.*

From now on we only consider polynomial-like mappings with only one branched point of degree d, assumed to be at zero by normalization, and refer to them as *unicritical polynomial-like* mappings. By the above theorem, any unicritical polynomial-like mapping with connected Julia set corresponds to a unique (up to affine conjugacy) unicritical polynomial $z \mapsto z^d + c$, with c in $\mathcal{M}_d$. Note that $z^d + c$ and $z^d + c/\lambda$ are conjugate via $z \mapsto \lambda z$ for every $(d-1)$-th root of unity λ.

Given a polynomial-like mapping $f : U' \to U$, we can consider the *fundamental annulus* $A = U \setminus U'$. It is not canonical because any choice of $V' \Subset V$ such that $f : V' \to V$ is a polynomial-like mapping with the same Julia set gives a different annulus. However, we can associate a real number, the *modulus* of f, to any polynomial-like mapping f as follows:

$$\mathrm{mod}(f) = \sup \mathrm{mod}(A),$$

where the supremum is taken over all possible fundamental annuli A of f.

It is easy to see that the hybrid conjugacy obtained in the straightening theorem is not unique. However, given a polynomial-like mapping, one can build a hybrid conjugacy as in the straightening Theorem with a uniform bound on its dilatation in terms of the modulus of the polynomial-like mapping. As this is essential in the rest of this work, we formulate it in the following proposition.

Proposition 2.6. *For every $\eta > 0$, there exists a constant $K > 0$, such that if f is a polynomial-like mapping with $mod(f) \geq \mu$, then one can choose a hybrid conjugacy as in the straightening theorem whose dilatation is bounded by K.*

3. Modified principal nest

3.1. **Yoccoz puzzle pieces.** Recall that for a parameter $c \in \mathcal{M}_d$ outside of the main hyperbolic component, P_c has a unique dividing fixed point α_c. The $q \geq 2$ external rays landing at α_c together with an arbitrary equipotential E^r cut the domain inside E^r into q closed *topological disks* (i.e. simply connected domains in $\mathbb{C}$) $Y_j^0, j = 0, 1, \ldots, q-1$, called *puzzle pieces of level zero*. That is, Y_j^0's are the closures of the bounded components of $\mathbb{C} \setminus \{E^r \cup \overline{\mathfrak{R}(\alpha_c)}\}$. The main property of this partition is that each $P_c(\partial Y_j^0)$ does not intersect the interior of any piece Y_i^0.

Puzzle pieces Y_i^n of level or depth n are defined as the closures of the connected components of $P_c^{-n}(\mathrm{int}(Y_j^0))$. They partition the region bounded by the equipotential $P_c^{-n}(E^r)$ into a finite number of closed disks. By definition, all puzzle pieces are bounded by piecewise analytic curves. The puzzle piece of level n containing the critical point is referred to as the *critical puzzle piece* of level n. The *label* of a puzzle piece is the set of the angles of the external rays bounding that puzzle piece. If the critical point does not land on α_c, there is a unique critical puzzle piece Y_0^n of every level n.

The family of all puzzle pieces of P_c has the following *Markov property*:

- Any two puzzle pieces are either disjoint or nested. In the latter case, the puzzle piece of higher level is contained in the puzzle piece of lower level.
- Image of any puzzle piece of level $n \geqslant 1$ is a puzzle piece of level $n-1$. Moreover, $P_c : Y_j^n \to Y_k^{n-1}$ is either d-to-1 branched covering, or univalent. This depends on whether Y_j^n contains the critical point or not.

On the first level, there are $d(q-1)+1$ puzzle pieces organized as follows. The critical piece Y_0^1; the $q-1$ (off critical) pieces attached to the fixed point α_c that are denoted by Y_i^1, for $i = 1, 2, \ldots, q-1$; and the symmetric ones attached to $P_c^{-1}(\alpha_c) \setminus \{\alpha_c\}$ that are denoted by Z_i^1, for $i = 1, 2, \ldots, (d-1)(q-1)$. Moreover, $P_c|Y_0^1$, d-to-1 covers Y_1^1, $P_c|Y_i^1$ univalently covers Y_{i+1}^1, for every $i = 1, \ldots, q-2$, and $P_c|Y_{q-1}^1$ univalently covers $Y_0^1 \cup \bigcup_{i=1}^{(d-1)(q-1)} Z_i^1$. Thus, $P_c^q(Y_0^1)$ truncated by $P_c^{-1}(E^r)$ is equal to the union of Y_0^1 and Z_i^1's.

From now on we assume that $P_c^n(0) \neq \alpha_c$, for all n. Therefore, the critical puzzle piece of every level is uniquely determined. As it will be apparent in a moment, this condition is always the case for the parameters we are interested in.

3.2. **Favorite nest and renormalization.** Given a puzzle piece V containing 0, let $R_V : \mathrm{Dom}\, R_V \subseteq V \to V$ denote *the first return map* to V. It is defined at every point z in V for which there exists a positive integer t with $P_c^t(z) \in \mathrm{int}\, V$. For every such z, $R_V(z)$ is defined as $P_c^t(z)$, where t is the smallest positive integer with $P_c^t(z) \in \mathrm{int}\, V$. The Markov property of the puzzle pieces implies that any component of $\mathrm{Dom}\, R_V$ is contained in V, and moreover, the restriction of this return map (P_c^t, for some t) to such a component is either a d-to-1, or 1-to-1 proper map onto V. The component of $\mathrm{Dom}\, R_V$ containing the critical point is called the *central component* of R_V. If image of the critical point under the first return map belongs to the central component, the return is called *central*.

The *first landing map* L_V to a puzzle piece $V \ni 0$ is defined at all points $z \in \mathbb{C}$ for which there exists an integer $t \geq 0$ with $P_c^t(z) \in \operatorname{int} V$. It is the identity on V, and it univalently maps each component of $\operatorname{Dom} L_V$ onto V.

Consider a puzzle piece $Q \ni 0$. If the orbit of the critical point returns back to Q under iterates of P_c, the central component $P \subset Q$ of R_Q is the pullback of Q by P_c^m along the orbit $0, P_c(0), \dots, P_c^m(0)$, where m is the first moment when the critical orbit enters $\operatorname{int} Q$. Hence, $P_c^m : P \to Q$ is a proper map of degree d. This puzzle piece P is called the *first child* of Q.

The *favorite child* Q' of Q is constructed as follows: Let $p > 0$ be the first moment when $R_Q^p(0) \in \operatorname{int}(Q \setminus P)$ (if it exists). Now, let $q > 0$ be the first moment (if it exists) when $R_Q^{p+q}(0) \in \operatorname{int} P$. In other words, $p + q$ is the moment of the first return back to P after the first escape of the critical point from P under iterates of R_Q. Now, Q' is defined as the pullback of Q under R_Q^{p+q} containing the critical point. The Markov property of the puzzle pieces implies that the map $R_Q^{p+q} = P_c^k : Q' \to Q$ (for an appropriate $k > 0$) is a proper map of degree d. The main property of the favorite child is that the image of the critical point under $P_c^k : Q' \to Q$ belongs to the first child P.

Let P_c be a unicritical polynomial with $q > 1$ external rays landing at its α-fixed point, and form the corresponding Yoccoz puzzle pieces introduced in Section 3.1. The map P_c is called *satellite renormalizable*, (also called *immediately renormalizable* by Douady and Hubbard) if

$$P_c^{lq}(0) \in Y_0^1, \quad \text{for } l = 0, 1, 2, \dots.$$

The map $P_c^q : Y_0^1 \to P_c^q(Y_0^1)$ is a proper branched covering of degree d. However, its domain is not compactly contained in its range. One can slightly enlarge Y_0^1 so that it is compactly contained in its range (see [Mil00a] for a detailed argument). Thus, P_c^q can be turned into a unicritical polynomial-like mapping. Note that the above condition on the orbit of the critical point implies that the corresponding little Julia set is connected.

If P_c is not satellite renormalizable, then there is the first positive integer k such that $P_c^{kq}(0)$ belongs to some Z_i^1. Define Q^1 as the pullback of this Z_i^1 under P_c^{kq} containing the critical point. By the above process we form the first child P^1 and the favorite child Q^2 of Q^1. Repeating the above process we obtain a (finite or infinite) nest of puzzle pieces

$$Q^1 \supset P^1 \supset Q^2 \supset P^2 \supset \cdots \supset Q^n \supset P^n \supset \cdots \tag{3.1}$$

where P^i is the first child of Q^i, and Q^{i+1} is the favorite child of Q^i.

The above nest is finite if and only if one of the following happens:

- The map P_c is combinatorially non-recurrent, that is, the critical point does not return to some critical puzzle piece.
- The orbit of the critical point does not escape some P^n under iterates of R_{Q^n}, or equivalently, the first return maps to all the critical puzzle pieces of level bigger than some n are central.

Combinatorial rigidity of the combinatorially non-recurrent parameters has been taken care of in [Mil00a]. In the latter case, $R_{Q^n} = P_c^k : P^n \to Q^n$ (for an appropriate k) is a unicritical polynomial-like mapping of degree d with P^n compactly contained in Q^n. The map P is called *primitively renormalizable* in this case. Note

that the corresponding little Julia set is connected because all the returns of the critical point to Q^n are central by definition.

A unicritical polynomial is called *renormalizable* if it is satellite or primitively renormalizable.

3.3. **Complex bounds and pseudo-conjugacies.** The general strategy, starting with Yoccoz's work on quadratics [Hub93], to prove the local connectivity of some Julia sets and rigidity of complex unicritical polynomials has been showing that every nest of puzzle pieces shrink to a point. To deal with non-renormalizable and combinatorially recurrent polynomials, the following *a priori* bounds property has been proved in [AKLS09].

Theorem 3.1. *There exists a constant $\delta > 0$ such that for every $\varepsilon > 0$ there exists $n_0 > 0$, with the following property. In the nest of puzzle pieces* (3.1)*, if* $\mathrm{mod}\,(Q^1 \setminus P^1) > \varepsilon$*, then for all $n \geq n_0$ we have* $\mathrm{mod}\,(Q^n \setminus P^n) > \delta$*.*

If P_c is combinatorially recurrent, the orbit of the critical point does not land at α-fixed point. Therefore, puzzle pieces of all levels are well defined. Now, let P_c be a non-renormalizable unicritical polynomial. The *combinatorics of* P_c up to level n is defined as an equivalence relation on the set of labels of all puzzle pieces of level less than or equal to n. Two such angles θ_1 and θ_2 are considered equivalent if R^{θ_1} and R^{θ_2} land at the same point. One can see that the combinatorics of a map up to level $n+t$ determines the puzzle piece Y_j^n of level n containing the critical value $P_c^t(0)$, for all positive integers n and t. Two non-renormalizable maps are called *combinatorially equivalent* if they have the same combinatorics up to an arbitrary level n. The combinatorics of a renormalizable map will be defined in Section 3.4.

Two unicritical polynomials P_c and $P_{\tilde{c}}$ with the same combinatorics up to some level n are called *pseudo-conjugate up to level* n if there exists an orientation preserving homeomorphism $H : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, such that $H(Y_j^0) = \widetilde{Y}_j^0$, for all j, and $H \circ P_c = P_{\tilde{c}} \circ H$ outside of the critical puzzle piece Y_0^n. Such a pseudo-conjugacy H is said to *match the Böttcher marking*, if near infinity it becomes the identity in the Böttcher coordinates for P_c and $P_{\tilde{c}}$. Thus, by the equivariance property of a pseudo-conjugacy, it is the identity map in the Böttcher coordinates outside of $\cup_j Y_j^n$.

Let q_m and p_m denote the levels of the puzzle pieces Q^m and P^m, respectively, in the nest (3.1), that is, $Q^m = Y_0^{q_m}$, and $P^m = Y_0^{p_m}$. The following theorem is the main technical result of [AKLS09] which is frequently used in the proof of our main theorem.

Theorem 3.2. *Assume that a nest of puzzle pieces*

$$Q^1 \supset P^1 \supset Q^2 \supset P^2 \supset \cdots \supset Q^m \supset P^m \tag{3.2}$$

is obtained for P_c, and $P_{\tilde{c}}$ is combinatorially equivalent to P_c up to level q_m, where $Q^m = Y_0^{q_m}$. Then there exists a K-q.c. pseudo-conjugacy H up to level q_m between P_c and $P_{\tilde{c}}$ which matches the Böttcher marking.

To control the dilatation of the pseudo-conjugacy obtained in this theorem, we show the following statement.

Proposition 3.3. *Assume that the nest of puzzle pieces in the above theorem is defined using the equipotential of level η, then the dilatation of the q.c. pseudo-conjugacy obtained in that theorem depends only on the hyperbolic distance between c and $\tilde{c}$ in the primary wake (containing c and $\tilde{c}$) truncated by the parameter equipotential of level η, and the modulus of the annulus $Q^1 \setminus P^1$.*

Proof. To prove the proposition, we need a brief sketch of the proof of the above theorem. For more details one may refer to [AKLS09].

The combinatorial equivalence of P_c and $P_{\tilde{c}}$ up to level zero implies that c and $\tilde{c}$ belong to the same truncated wake $W(\eta)$ attached to the main hyperbolic component of $\mathcal{M}_d$. Inside $W(\eta)$, the q external rays $\mathfrak{R}(\alpha)$ and E^h, for any $h > \eta$, move holomorphically in $\mathbb{C} \setminus 0$. That is, there exists a holomorphic motion $\Phi : W(\eta) \times \{\mathfrak{R}(\alpha) \cup E^h\} \to W(\eta) \times \mathbb{C}$, given by $B_{\tilde{c}}^{-1} \circ B_c$ in the second coordinate, such that

$$\Phi(c, \mathfrak{R}(\alpha) \cup E^h) = (\tilde{c}, \mathfrak{R}(\widetilde{\alpha}) \cup \widetilde{E}^h).$$

Outside of the equipotential E^h, this holomorphic motion extends to the motion holomorphic in both variables (c, z) which is obtained from the Böttcher coordinates near ∞. By [Slo91] the map $\Phi(\tilde{c}, \cdot) \circ \Phi(c, \cdot)^{-1}$ extends to a K_0-q.c. homeomorphism $G_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, where K_0 depends only on the hyperbolic distance between c and $\tilde{c}$ in $W(\eta)$. The map G_0 conjugates P_c to $P_{\tilde{c}}$ outside of the puzzle pieces of level zero.

By adjusting G_0 inside the equipotential E^h such that it sends c to $\tilde{c}$, we obtain a q.c. homeomorphism (not necessarily with the same dilatation) G_0'. By lifting G_0' via P_c and $P_{\tilde{c}}$ we obtain a new q.c. homeomorphism G_1. That is, G_1 is the unique map satisfying $P_{\tilde{c}} \circ G_1 = G_0' \circ P_c$.

Now, we repeat the following two processes, for $i = 2, 3, \ldots, n = q_m$,

- Adjust the q.c. homeomorphism G_{i-1} inside the union of puzzle pieces of level i so that it sends c to $\tilde{c}$;
- Lift the adjusted map via P_c and $P_{\tilde{c}}$ to obtain a q.c. homeomorphism G_i (not necessarily with the same dilatation) conjugating P_c to $P_{\tilde{c}}$ outside the union of puzzle pieces of level i.

At the end, we obtain a q.c. homeomorphism G_n which conjugates P_c to $P_{\tilde{c}}$ outside of the equipotential E^{h/d^n}.

The nest of puzzle pieces

$$\widetilde{Q}^1 \supset \widetilde{P}^1 \supset \widetilde{Q}^2 \supset \widetilde{P}^2 \supset \cdots \supset \widetilde{Q}^m \supseteq \widetilde{P}^m$$

for $P_{\tilde{c}}$ is defined as the image of the nest (3.2) under G_n. The combinatorial equivalence of P_c and $P_{\tilde{c}}$ implies that this new nest has the same properties as the one of P_c. In other words, $\widetilde{Q}^{i+1}$ is the favorite child of $\widetilde{Q}^i$, and $\widetilde{P}^i$ is the first child of $\widetilde{Q}^i$. Hence, Theorem 3.1 applies to this nest as well. By properties of these nests, one constructs a K-q.c. homeomorphism H_n from the critical puzzle piece Q^n to $\widetilde{Q}^n$, where K depends only on the *a priori* bounds δ and the hyperbolic distance between c and $\tilde{c}$ in $W(\eta)$. The pseudo-conjugacy H_n is obtained from univalent lifts of H_n onto other puzzle pieces. □

If P_c is renormalizable, the process of constructing the modified principal nest stops at some level, and all the returns to the critical puzzle pieces of higher level become central. One can see that in this situation the critical puzzle pieces do not shrink to 0.

3.4. **Combinatorics of a map.** If a map P_{c_0} is renormalizable, there is a unique maximal homeomorphic copy $\mathcal{M}_d^1 \ni c_0$ of the connectedness locus within the connectedness locus satisfying the following properties (see[DH85b]): For $c \in \mathcal{M}_d^1 \setminus \{\text{root}\}$, $P_c : z \mapsto z^d + c$ is renormalizable, and in addition, there is a holomorphic motion of the dividing fixed point α_c and the rays landing at it on a neighborhood of $\mathcal{M}_d^1 \setminus \{\text{root}\}$, such that the renormalization of P_c is associated to this fixed point and the external rays landing at it. Furthermore, all parameters in this copy have Yoccoz puzzle pieces of all levels with the same labels. This copy is maximal in the sense that it is not contained in any other copy except the actual connectedness locus. The homeomorphism from the copy to $\mathcal{M}_d$ is not unique because of the rotational symmetry of $\mathcal{M}_d$. However, we make it unique by sending the sole root point of the copy to the landing point of the parameter external ray of angle zero. We denote this (first) renormalization of P_c by $\mathcal{R}P_c$.

Assume that $\mathcal{R}P_c$ is given as $P_c^j : U \to U'$, for some positive integer j and topological disks U and U'. By the straightening theorem, $\mathcal{R}P_c$ is conjugate to a unicritical polynomial $P_{c'}$. The polynomial $P_{c'}$ is determined up to conformal equivalence in this theorem. However, there are only $d-1$ polynomials in each conformal class (these are $c' \cdot \lambda$, for λ with $\lambda^{d-1} = 1$). We make this parameter unique by choosing the image of c under the unique homeomorphism from the copy to the connectedness locus determined above.

If $P_{c'}$ is also renormalizable, P_c is called *twice renormalizable*. Let the positive integer k, and topological disks V and V' be such that $P_{c'}^k : V \to V'$ gives the first renormalization of $P_{c'}$ determined as above. Define the topological disks $\widetilde{V}$ and $\widetilde{V}'$ as χ-preimages of V and V', respectively, where χ is a straightening of $\mathcal{R}P_c$. One can see that χ conjugates $P_c^{jk} : \widetilde{V} \to \widetilde{V}'$ to $P_{c'}^k : V \to V'$. Therefore, $P_c^{jk} : \widetilde{V} \to \widetilde{V}'$ is also polynomial-like. We denote this map by $\mathcal{R}^2 P_c$.

The above process may be continued to associate a finite or an infinite sequence $P_c, \mathcal{R}P_c, \mathcal{R}^2 P_c, \ldots$, of polynomial-like mappings to P_c, and accordingly, call P_c *at most finitely* or *infinitely renormalizable*. Let $P_{c_1}, P_{c_2}, P_{c_3}, \ldots$ denote the polynomials obtained from straightening the polynomial-like mappings $\langle \mathcal{R}^n P_c \rangle_{n=0}$. We associate the finite or infinite sequence

$$\tau(P_c) := \langle \mathcal{M}_d^1, \mathcal{M}_d^2, \ldots \rangle,$$

of the maximal copies of the locus to P_c, where $\mathcal{M}_d^n$ corresponds to the renormalization $\mathcal{R}P_{c_{n-1}}$. Earlier in Section 3.3 we defined the combinatorics of a non-renormalizable unicritical polynomial as the equivalence relation on the labels of the Yoccoz puzzle pieces. It turns out that all the parameters in a given copy of the connectedness locus within the parameter space have the same combinatorics in this sense. To further refine our definition of the combinatorics, one may consider the same equivalence relation, i.e. landing at the same point, on a larger set of angles of external rays. Here, for an infinitely renormalizable P_c, the sequence $\tau(P_c)$ is called the *combinatorics* of P_c. This definition, which is chosen for our convenience, is equivalent to the above definition of the combinatorics when we consider the equivalence relation on the set of angles of all periodic external rays.

Hence, two infinitely renormalizable maps are called *combinatorially equivalent* if they have the same combinatorics, i.e., correspond to the same sequence of maximal connectedness locus copies.

We say that an infinitely renormalizable P_c satisfies the *secondary limbs condition*, if all the parameters $c_1, c_2, \ldots$, obtained from straightening the polynomial-like mappings $\{\mathcal{R}^n P_c\}_{n=0}^{\infty}$ belong to a finite number of truncated secondary limbs. Let $\mathcal{SL}$ stand for the class of infinitely renormalizable unicritical polynomial-like mappings satisfying the secondary limbs condition.

An infinitely renormalizable map P_c is said to satisfy *a priori bounds*, if there exists an $\varepsilon > 0$ with $\operatorname{mod}(\mathcal{R}^m P_c) \geq \varepsilon$, for all $m \geq 1$.

4. Proof of the rigidity theorem

4.1. **Reductions.**

Theorem 4.1. *Let f and $\tilde{f}$ be two infinitely renormalizable unicritical polynomial-like mappings satisfying the $\mathcal{SL}$ and a priori bounds conditions. If f and $\tilde{f}$ are combinatorially equivalent, then they are hybrid equivalent.*

Remark 4.2. In particular, if the two maps f and $\tilde{f}$ in the above theorem are polynomials, then hybrid equivalence becomes conformal equivalence. That is because the identity map in the Böttcher coordinates, which conformally conjugates the two maps on the complements of their Julia sets, can be glued to the hybrid conjugacy on the Julia set. See Proposition 6 in [DH85b] for a precise proof of this statement.

The proof of this theorem breaks into the following steps:

combinatorial equivalence

$\Downarrow$

topological equivalence

$\Downarrow$

q.c. equivalence

$\Downarrow$

hybrid equivalence

It has been shown in [Jia00] that any *unbranched* infinitely renormalizable map with *a priori* bounds has a locally connected Julia set. The renormalizations $f^{n_k} : U_k \to V_k$, for $k = 1, 2, 3, \ldots$, are said to satisfy the unbranched property if the domains U_k and V_k which provide the *a priori* bounds also satisfy

$$\mathcal{PC}(f) \cap U_k = \mathcal{PC}(f^{n_k} : U_k \to V_k), \text{ for } k = 1, 2, 3, \ldots.$$

Here, the unbranched property follows from our combinatorial and *a priori* bounds conditions (see [Lyu97], Lemma 9.3). Then, the first step, topological equivalence of combinatorially equivalent maps, follows from the local connectivity of the Julia sets by Carathéodory's Theorem. That is, the identity map in the Böttcher coordinates extends onto the Julia set. Indeed, by [Dou93], there is a topological model for the Julia sets of these maps based on their combinatorics.

The last step follows from McMullen's rigidity Theorem ([McM94], Theorem 10.2). He has shown that an infinitely renormalizable quadratic polynomial-like mapping with *a priori* bounds does not support any nontrivial invariant line fields on its Julia set. The same proof works for unicritical polynomial-like mappings of any degree. It follows that any q.c. conjugacy h between f and $\tilde{f}$ satisfies $\overline{\partial} h = 0$ almost everywhere on the Julia set. Therefore, h is a hybrid conjugacy between f and $\tilde{f}$. However, if all infinitely renormalizable unicritical maps in a

given combinatorial class satisfy the *a priori* bounds condition, there is an easier way to show that q.c. conjugate maps are hybrid conjugate in that class. Since we are going to apply our theorem to combinatorial classes for which *a priori* bounds has been established, we will prove this in Proposition 4.17.

So assume that f and $\tilde{f}$ are topologically conjugate. We want to show the following:

Theorem 4.3. *Let f and $\tilde{f}$ be infinitely renormalizable unicritical polynomial-like mappings satisfying the a priori bounds and $\mathcal{SL}$ conditions. If f and $\tilde{f}$ are topologically conjugate then they are quasi-conformally conjugate.*

Given sets $A \subseteq B \subseteq C$ and $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$, the notation $h : (C, B, A) \to (\widetilde{C}, \widetilde{B}, \widetilde{A})$ means that h is a map from C to $\widetilde{C}$ with $h(B) = \widetilde{B}$ and $h(A) = \widetilde{A}$.

4.2. **Thurston equivalence.** Suppose that two unicritical polynomial mappings $f : U_2 \to U_1$ and $\tilde{f} : \widetilde{U}_2 \to \widetilde{U}_1$ are topologically conjugate. A q.c. homeomorphism

$$h : (U_1, U_2, \mathcal{PC}(f)) \to (\widetilde{U}_1, \widetilde{U}_2, \mathcal{PC}(\tilde{f}))$$

is called a *Thurston conjugacy* if it is homotopic, relative $\partial U_1 \cup \partial U_2 \cup \mathcal{PC}(f)$, to a topological conjugacy

$$\psi : (U_1, U_2, \mathcal{PC}(f)) \to (\widetilde{U}_1, \widetilde{U}_2, \mathcal{PC}(\tilde{f}))$$

between f and $\tilde{f}$. Note that a Thurston conjugacy does not conjugate the two maps. It is a conjugacy on the post-critical set, and homotopic to a conjugacy on the complement of the post-critical set.

The following result due to Thurston and Sullivan [Sul92] originates the "pull-back method" in holomorphic dynamics.

Lemma 4.4. *Thurston conjugate unicritical polynomial-like mappings are q.c. conjugate.*

The proof given in [Lyu97], Lemma 10.1, in the quadratic case works for the unicritical maps without any change.

By the *a priori* bounds assumption in the theorem, there are topological disks $V_{n,0} \Subset U_{n,0}$ containing zero such that $\mathcal{R}^n f := f^{t_n} : V_{n,0} \to U_{n,0}$, for $n = 1, 2, \dots$, are unicritical degree d polynomial-like with $\operatorname{mod}(U_{n,0} \setminus V_{n,0}) \geq \varepsilon$, for some $\varepsilon > 0$. By going several levels down, i.e., considering $f^{t_n} : f^{-k_n t_n}(V_n) \to f^{-k_n t_n}(U_n)$, for some positive integers k_n, we may assume that $\operatorname{mod}(U_n \setminus V_n)$ and $\operatorname{mod}(\widetilde{U}_n \setminus \widetilde{V}_n)$ are uniformly bounded from above and are *comparable*. The latter means that, there exists a constant $M > 0$ such that for every $n \geq 1$,

$$\frac{1}{M} \leq \frac{\operatorname{mod}(U_n \setminus V_n)}{\operatorname{mod}(\widetilde{U}_n \setminus \widetilde{V}_n)} \leq M. \tag{4.1}$$

Also by slightly shrinking the domains $U_{n,0}$, if necessary, we may assume that they have smooth boundaries. Hence, we can assume the following in the remainder of this note.

- There exist positive constants ε and η such that for every $n \geq 1$,

$$\varepsilon \leq \operatorname{mod}(U_{n,0} \setminus V_{n,0}) \leq \eta, \text{ and } \varepsilon \leq \operatorname{mod}(\widetilde{U}_{n,0} \setminus \widetilde{V}_{n,0}) \leq \eta; \tag{4.2}$$

- For every $n \geq 1$, $U_{n,0}$ and $\widetilde{U}_{n,0}$ have smooth boundaries;
- There exists a constant $M > 0$, for which (4.1) holds for every $n \geq 1$.

We use the following notations throughout the rest of this note.

$$f : V_0 \to U_0, J_{0,0} = J(f),$$
$$\mathcal{R}^n f = f^{t_n} : V_{n,0} \to U_{n,0}, J_{n,0} = J(\mathcal{R}^n f), \text{ for } n \geq 1 \text{ and } t_n \geq 1.$$

The domain $V_{n,i}$, for $i = 1, 2, \ldots, t_n - 1$, is defined as the preimage of $V_{n,0}$ under f^i containing the *little Julia set* $J_{n,i} := f^{t_n - i}(J_{n,0})$. Similarly, $U_{n,i}$, for $i = 1, 2, \ldots, t_n - 1$, is defined as the component of $f^{-i}(U_{n,0})$ containing $V_{n,i}$ so that $f^{t_n} : V_{n,i} \to U_{n,i}$ is polynomial-like of degree d. The domain $W_{n,i}$ is defined as the preimage of $V_{n,i}$ under the map $f^{t_n} : V_{n,i} \to U_{n,i}$.

Note that $\mathcal{R}^n f : V_{n,i} \to U_{n,i}$ is a polynomial-like mapping of degree d with the Julia set $J_{n,i}$, and is conjugate to $\mathcal{R}^n f : V_{n,0} \to U_{n,0}$ by the conformal isomorphism $f^i : U_{n,i} \to U_{n,0}$.

It has been proved in [Lyu97] (Lemma 9.2) that for the parameters satisfying our assumptions, the little Julia sets on the primitive levels are *well apart*. This means that for every $n \geq 1$ with $\mathcal{R}^{n-1} f$ primitively renormalizable, one can choose pairwise disjoint domains $U_{n,i}$, for $i = 1, 2, \ldots, t_n - 1$, for the renormalizations with moduli of the annuli $U_{n,i} \setminus V_{n,i}$ uniformly away from zero independent of n and i. So, we will assume that on the primitive levels, the domains $U_{n,i}$ are disjoint for different values of i.

Remark on the notations 1. From now on, any notation introduced for f will be automatically introduced for $\tilde{f}$, and marked with a tilde.

To build a Thurston conjugacy, first we introduce multiply connected domains $\Omega_{n(k),i}$ (and $\widetilde{\Omega}_{n(k),i}$) in $\mathbb{C}$, for an appropriate subsequence $n(1) < n(2) < n(3) < \cdots$ of the renormalization levels and $0 \leq i \leq t_{n(k)} - 1$, as well as a sequence of q.c. homeomorphisms

$$h_{n(k),i} : \Omega_{n(k),i} \to \widetilde{\Omega}_{n(k),i},$$

for $k = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots, t_{n(k)} - 1$, with uniformly bounded dilatations. These domains will satisfy the following properties (see Figure 4):

- Every $\Omega_{n(k),j}$ is a topological disk minus $\frac{t_{n(k+1)}}{t_{n(k)}}$ number of topological disks denoted by $D_{n(k+1),j+it_{n(k)}}$, for $i = 0, 1, \ldots, \frac{t_{n(k+1)}}{t_{n(k)}} - 1$;
- Every $\Omega_{n(k),i}$ is *well inside* $D_{n(k),i}$, that is, the moduli of the annuli obtained from $D_{n(k),i} \setminus \Omega_{n(k),i}$ are uniformly bounded away from zero independent of $n(k)$ and i;
- Every *little post-critical set* $J_{n(k),i} \cap \mathcal{PC}(f)$ is well inside $D_{n(k),i}$;
- Every $D_{n(k),i}$ is the preimage of $D_{n(k),0}$ under f^i containing $J_{n(k),i} \cap \mathcal{PC}(f)$. Every $\Omega_{n(k),i}$ is the component of $f^{-i}(\Omega_{n(k),0})$ inside $D_{n(k),i}$.

Finally, we construct the Thurston conjugacy by appropriately gluing together the maps $h_{n(k),i} : \Omega_{n(k),i} \to \widetilde{\Omega}_{n(k),i}$ on the complement of these multiply connected domains (which is a union of annuli).

4.3. **The domains $\Omega_{n,j}$ and the maps $h_{n,j}$.** By straightening the polynomial-like mappings

$$\mathcal{R}^{n-1} f : V_{n-1,0} \to U_{n-1,0},$$

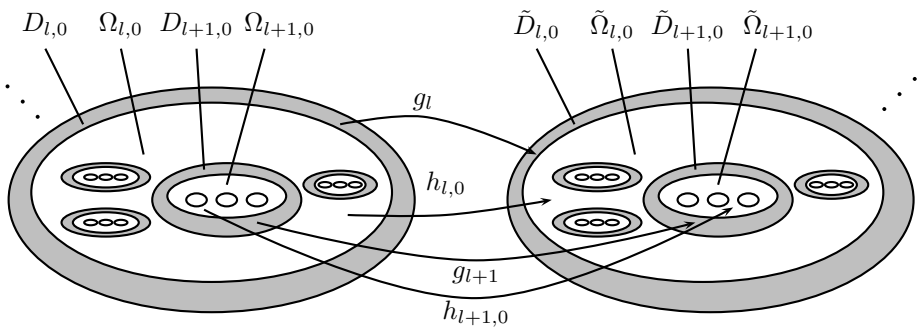

FIGURE 4. The multiply connected domains and the buffers

for $n = 2, 3, 4, \ldots$, we get $K_1(\varepsilon)$-q.c. homeomorphisms S_{n-1} as well as unicritical polynomials $\mathbf{f}_{c_{n-1}}$, such that

$$S_{n-1} : (U_{n-1,0}, V_{n-1,0}, 0) \to (\Upsilon^0_{n-1}, \Upsilon^1_{n-1}, 0), \textit{and} \tag{4.3}$$
$$S_{n-1} \circ \mathcal{R}^{n-1} f = \mathbf{f}_{c_{n-1}} \circ S_{n-1}.$$

Note that $\mathbf{f}_{c_{n-1}}$ is made unique by the argument in Section 3.4. See Figure 5.

Remark on the notations 2. To make our notations easier to follow, we drop the second subscripts whenever they are zero and do not create any confusion. Also, all objects on the dynamic planes of $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$ (the ones after the straightenings) will be denoted by the boldface of the notations used for the corresponding objects on the dynamic planes of f and $\tilde{f}$.

To define $\Omega_{n-1,j}$ and $h_{n-1,j}$, we need to consider the following three cases:

$\mathscr{A}$. $\mathcal{R}^{n-1} f$ is primitively renormalizable;
$\mathscr{B}$. $\mathcal{R}^{n-1} f$ is satellite renormalizable and $\mathcal{R}^n f$ is primitively renormalizable;
$\mathscr{C}$. both $\mathcal{R}^{n-1} f$ and $\mathcal{R}^n f$ are satellite renormalizable.

We introduce $\Omega_{n-1,j}$ and $h_{n-1,j}$ in each of the above cases. Then one consecutively applies these cases to construct all the domains $\Omega_{n-1,j}$ and the maps $h_{n-1,j}$. In what follows, to explain how we choose these cases consecutively, we associate a string of cases to f, depending only on its combinatorics. More precisely, for a string of cases $\mathscr{A}^{m_1}\mathscr{B}^{m_2}\mathscr{C}^{m_3}\ldots$, with non-negative integers m_j, obtained for f, we repeat Case $\mathscr{A}$, m_1 times then repeat Case $\mathscr{B}$, m_2 times and so on.

Given f_c, its renormalization on each level is of primitive or satellite type. Therefore, we can associate the string

$$P \ldots PS \ldots SP \ldots \tag{4.4}$$

in P and S, where P or S in the i-th place means that the i-th renormalization of f_c is of primitive or satellite type, respectively. Corresponding to any such string, we define a string of cases $\mathscr{A}^{m_1}\mathscr{B}^{m_2}\mathscr{C}^{m_3}\ldots$, with non-negative integers m_j, as follows. Inductively, starting from the left, P is replaced by $\mathscr{A}$, SP by $\mathscr{B}$, and SS

by $\mathscr{C}S$. Repeating this process, we obtain the desired string of cases. This also introduces the sequence $n(k)$, for $k = 1, 2, 3, \dots$ as follows. Given the string (4.4), the sequence $n(k)$ is obtained from the sequence of natural numbers $1, 2, 3, \dots$ by removing all the integers l for which there is an S in the $(l-1)$-th place and a P in the l-th place. That is, we skip the level of any primitive renormalization occurring after a satellite one. For example:

$$\begin{aligned} \text{A string of renormalizations: } & PSPPSSPSSSPP\dots \\ \text{Its string of cases: } & \mathscr{A}\mathscr{B}\mathscr{A}\mathscr{C}\mathscr{B}\mathscr{C}\mathscr{C}\mathscr{B}\mathscr{A}\dots \\ \text{Its sequence: } & 1, 2, 4, 5, 6, 8, 9, 10, 12, \dots \end{aligned}$$

The following lemma shows that there are equipotentials of sufficiently high level $\eta(\varepsilon)$ inside $S_{n-1}(W_{n-1,0})$ and $\widetilde{S}_{n-1}(\tilde{W}_{n-1,0})$ in the dynamic planes of the maps $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$.

Lemma 4.5. *For every $\varepsilon > 0$ and positive integer d, there exists $\eta > 0$ such that if $P_c : U' \to U$ is a proper unicritical polynomial of degree d with connected Julia set and $\operatorname{mod}(U \setminus U') \geq \varepsilon$, then U' contains equipotentials of level less than η.*

Proof. The map P_c on the complement of $K(P_c)$ is conjugate to the map P_0 on the complement of the closed unit disk D_1 by the Böttcher coordinate B_c. Since levels of the equipotentials are preserved under this map, and modulus is a conformal invariant, it is enough to prove the lemma for $P_0 : V' \to V$, with V' compactly contained in V and $\operatorname{mod}(V \setminus V') \geq \varepsilon$. As $P_0 : P_0^{-1}(V \setminus V') \to (V \setminus V')$ is a covering of degree d, the modulus of the annulus $P_0^{-1}(V \setminus V')$ is at least ε/d. Hence, $\operatorname{mod}(V' \setminus D_1)$ is at least ε/d. By the Grötzsch problem in [Ahl06] (Section A in Chapter III) we conclude that $V' \setminus D_1$ must contain a round annulus $D_\eta \setminus D_1$. □

Case $\mathscr{A}$: By considering equipotentials of level $\eta(\varepsilon)$ contained in S_{n-1} of $W_{n-1,0}$ and $\widetilde{S}_{n-1}$ of $\tilde{W}_{n-1,0}$, obtained in the previous lemma, and the external rays landing at the dividing fixed points α_{n-1} and $\tilde{\alpha}_{n-1}$ of the maps $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$, we form the favorite nest of puzzle pieces (3.1) introduced in Section 3.2.

Let $Q_{n,0}^{\chi_n} := Y_{n,0}^{q_{\chi_n}}$ and $P_{n,0}^{\chi_n}$ denote the last critical puzzle pieces obtained in the nest (3.1), i.e. χ_n is the smallest positive integer with $\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : P_n^{\chi_n} \to Q_n^{\chi_n}$ is a unicritical polynomial-like mapping of degree d with connected Julia set. Now, let

$$\mathbf{h}_{n-1} = \mathbf{h}_{n-1,0} : \mathbb{C} \to \mathbb{C},$$

denote the K_2-q.c. pseudo-conjugacy one obtains from Theorem 3.2. The distance between the parameters c_{n-1} and $\tilde{c}_{n-1}$ with respect to the hyperbolic metric on the truncated primary wake containing c_{n-1} and $\tilde{c}_{n-1}$, $W(\eta(\varepsilon))$, is bounded by some M depending only on the combinatorial class $\mathcal{SL}$. That is because c_{n-1} and $\tilde{c}_{n-1}$ belong to a compact subset of $W(\eta(\varepsilon))$ consisting of a finite number of truncated limbs. It has been proved in [Lyu97], Theorem I, that $\operatorname{mod}(Q^1 \setminus P^1)$ is uniformly bounded away from zero depending only on the combinatorial class $\mathcal{SL}$. The same proof, based on the continuous dependence of certain rays on the parameter and compactness of the class $\mathcal{SL}$, works for the unicritical polynomials as well. Therefore, by Proposition 3.3, $\operatorname{Dil}(\mathbf{h}_{n-1})$ is uniformly bounded by some constant K_2 depending only on ε and the class $\mathcal{SL}$.

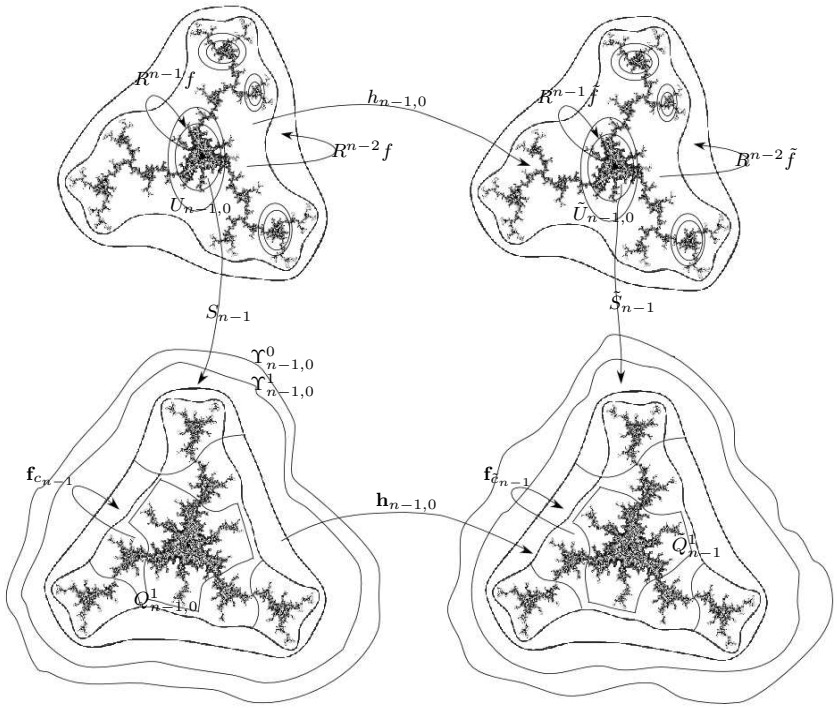

FIGURE 5. Primitive case

For $i = 0, 1, 2, \ldots, t_n/t_{n-1} - 1$, denote the components of the sets $\mathbf{f}_{c_{n-1}}^{-i}(Q^{\chi_n}_{n,0})$ and $\mathbf{f}_{c_{n-1}}^{-i}(P^{\chi_n}_{n,0})$ containing

$$\mathbf{J}_{1,i} := \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}-i}(J(\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : P^{\chi_n}_n \to Q^{\chi_n}_n),$$

by $Q^{\chi_n}_{n,i}$ and $P^{\chi_n}_{n,i}$, respectively. Note that t_n/t_{n-1} is the period of the first renormalization of $\mathbf{f}_{c_{n-1}}$.

The polynomials $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$ also satisfy our combinatorial and *a priori* bounds assumptions. Therefore, there is a topological conjugacy $\boldsymbol{\psi}_{n-1}$ between them, obtained from extending $B^{-1}_{\tilde{c}_{n-1}} \circ B_{c_{n-1}}$ onto $J(\mathbf{f}_{c_{n-1}})$.

Now, we adjust $\mathbf{h}_{n-1} : Q^{\chi_n}_n \to \widetilde{Q}^{\chi_n}_n$, using the dynamics of the maps

$$\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : P^{\chi_n}_n \to Q^{\chi_n}_n \quad \text{and} \quad \mathbf{f}_{\tilde{c}_{n-1}}^{t_n/t_{n-1}} : \widetilde{P}^{\chi_n}_n \to \widetilde{Q}^{\chi_n}_n,$$

so that the equivariance property (for $\mathbf{h}_{n-1}$) holds on a larger set. Let A^0_n denote the closure of the annulus $Q^{\chi_n}_n \setminus P^{\chi_n}_n$, and A^k_n, for $k = 1, 2, 3, \ldots$, denote the component of $\mathbf{f}_{c_{n-1}}^{-kt_n/t_{n-1}}(A^0_n)$ around $J_{n,0}$. Then, we lift $\mathbf{h}_{n-1}$ via $\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : Q^{\chi_n}_n \to \mathbb{C}$ and $\mathbf{f}_{\tilde{c}_{n-1}}^{t_n/t_{n-1}} : \widetilde{Q}^{\chi_n}_n \to \mathbb{C}$ to obtain a K_2-q.c. homeomorphism $g : A^0_n \to \widetilde{A}^0_n$ which is homotopic to $\boldsymbol{\psi}_{n-1}$ relative ∂A^0_n. That is because the external rays connecting $\partial P^{\chi_n}_n$ to $\partial Q^{\chi_n}_n$, partition A^0_n into several topological disks, and the two maps $\boldsymbol{\psi}_{n-1}$ and g coincide on the boundaries of these topological disks.

As $\mathbf{f}_{c_{n-1}}^{kt_n/t_{n-1}} : A_n^k \to A_n^0$ and $\mathbf{f}_{\tilde{c}_{n-1}}^{kt_n/t_{n-1}} : \widetilde{A}_n^k \to \widetilde{A}_n^0$, for every $k \geq 1$, are holomorphic unbranched coverings, g can be lifted to a K_2-q.c. homeomorphism from A_n^k to $\widetilde{A}_n^k$. All these lifts are the identity in the Böttcher coordinates on the boundaries of these annuli. Hence, they glue together to form a q.c. homeomorphism $g : Q_n^{\chi_n} \setminus \mathbf{J}_{1,0} \to \tilde{Q}_n^{\chi_n} \setminus \tilde{\mathbf{J}}_{1,0}$ which conjugate

$$\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : P_n^{\chi_n} \setminus \mathbf{J}_{1,0} \to Q_n^{\chi_n} \setminus \mathbf{J}_{1,0}, \text{ to } \mathbf{f}_{\tilde{c}_{n-1}}^{t_n/t_{n-1}} : \widetilde{P}_n^{\chi_n} \setminus \widetilde{\mathbf{J}}_{1,0} \to \widetilde{Q}_n^{\chi_n} \setminus \widetilde{\mathbf{J}}_{1,0}.$$

Finally, we would like to extend this map (as a homeomorphism) onto $\mathbf{J}_{1,0}$. This is a special case of a more general argument presented below.

Given a polynomial f with connected filled Julia set $K(f)$, the *rotation* of angle θ on $\mathbb{C} \setminus K(f)$ is defined as the rotation of angle θ in the Böttcher coordinate on $\mathbb{C} \setminus K(f)$, that is, $B_c^{-1}(e^{\mathbf{i}\theta} \cdot B_c)$. By means of straightening, one can define rotations on the complement of the filled Julia set of a polynomial-like mapping. It is not canonical as it depends on the choice of the straightening. However, its effect on the landing points of external rays is canonical.

Proposition 4.6. *Let $f : V_2 \to V_1$ be a polynomial-like mapping with connected filled Julia set $K(f)$. If $\phi : V_1 \setminus K(f) \to V_1 \setminus K(f)$ is a homeomorphism which commutes with f, then there exists a rotation of angle $2\pi j/(d-1)$ for some j, denoted by ρ_j, such that $\rho_j \circ \phi$ extends onto $K(f)$ as the identity.*

A proof of this proposition is given in the Appendix.

Applying the above proposition to $\boldsymbol{\psi}_{n-1}^{-1} \circ g$ with $V_1 = Q_n^{\chi_n}$, $V_2 = P_n^{\chi_n}$, and an external ray connecting $\partial Q_n^{\chi_n}$ to $\mathbf{J}_{1,0}$, we conclude that g extends onto $\mathbf{J}_{1,0}$ as $\boldsymbol{\psi}_{n-1}$. Also, it follows from the proof of the above proposition that g and $\boldsymbol{\psi}_{n-1}$ are homotopic relative $\mathbf{J}_{1,0} \cup \partial Q_n^{\chi_n}$. That is because the quadrilaterals obtained in the proof cut the puzzle piece $Q_n^{\chi_n}$ into infinite number of topological disks such that g and $\boldsymbol{\psi}_{n-1}$ are equal on the boundaries of these topological disks.

Similarly, $\mathbf{h}_{n-1}$ can be adjusted on the other puzzle pieces $Q_{n,i}^{\chi_n}$, for $i = 1, 2, \ldots, t_n/t_{n-1}$, so that it is homotopic to $\boldsymbol{\psi}_{n-1}$ restricted to $Q_{n,i}^{\chi_n}$, relative $\mathbf{J}_{1,i} \cup \partial Q_{n,i}^{\chi_n}$. We denote the map obtained from extending $\mathbf{h}_{n-1}$ onto little Julia sets $\mathbf{J}_{1,i}$ with the same notation $\mathbf{h}_{n-1}$.

Finally, we need to prepare $\mathbf{h}_{n-1}$ for the next step (case) of the process. It is stated in the following lemma.

Lemma 4.7. *The K_2-q.c. homeomorphism $\mathbf{h}_{n-1}$ can be adjusted on a neighborhood of $\cup_i \mathbf{J}_{1,i}$, through a homotopy relative $\cup_i \mathbf{J}_{1,i}$, to a q.c. homeomorphism $\mathbf{h}'_{n-1}$ which maps $\mathbf{V}_{n,i} := S_{n-1}(V_{n,i})$ onto $\widetilde{\mathbf{V}}_{n,i} := \widetilde{S}_{n-1}(\widetilde{V}_{n,i})$. Moreover, $\mathrm{Dil}\,(\mathbf{h}'_{n-1})$ is uniformly bounded by a constant K_3 depending only on ε.*

Proof. The basic idea is to continuously move the domain $\mathbf{h}_{n-1}(\mathbf{V}_{n,i})$ close enough to $J_{1,i}$, and then move it back to $\widetilde{\mathbf{V}}_{n,i}$ (simultaneously for all $i = 0, 1, \ldots, t_n/t_{n-1}$). We will do this more precisely below. Let $\mathbf{U}_{n,i}$ denote $S_{n-1}(U_{n,i})$ and $\widetilde{\mathbf{U}}_{n,i}$ denote $\widetilde{S}_{n-1}(\widetilde{U}_{n,i})$.

The annuli $\mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \setminus \widetilde{\mathbf{J}}_{1,i}$ and $\widetilde{\mathbf{V}}_{n,i} \setminus \widetilde{\mathbf{J}}_{1,i}$ have moduli bigger than ε/dK_1K_2, where $K_1 := \mathrm{Dil}\,(S_{n-1})$ and $K_2 := \mathrm{Dil}\,(\mathbf{h}_{n-1})$. Therefore, there exist topological disks $\widetilde{\mathbf{L}}_{n,i} \supseteq \widetilde{\mathbf{J}}_{1,i}$, with smooth boundaries, and a constant $r > 0$ satisfying the following properties

- $\widetilde{\mathbf{L}}_{n,i} \subset \mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \cap \widetilde{\mathbf{V}}_{n,i}$,

– $\mod(\widetilde{\mathbf{L}}_{n,i} \setminus \widetilde{\mathbf{J}}_{1,i}) \geq r$,
– $\mod(\widetilde{\mathbf{V}}_{n,i} \setminus \widetilde{\mathbf{L}}_{n,i}) \geq \varepsilon/2dK_1K_2$, and
– $\mod(\mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \setminus \widetilde{\mathbf{L}}_{n,i}) \geq \varepsilon/2dK_1K_2$.

Now, there are q.c. homeomorphisms

$$\chi_i : \big(\mathbf{h}_{n-1}(\mathbf{U}_{n,i}), \mathbf{h}_{n-1}(\mathbf{V}_{n,i}), \widetilde{\mathbf{L}}_{n,i}, \mathbf{J}_{1,i}\big) \to \big(D_5, D_3, D_2, D_1\big),$$

with uniformly bounded dilatations. That is because the annuli

$$\widetilde{\mathbf{L}}_{n,i} \setminus \widetilde{\mathbf{J}}_{1,i}, \mathbf{h}_{n-1}(\mathbf{V}_{n,i}) \setminus \widetilde{\mathbf{L}}_{n,i}, \text{ and } \mathbf{h}_{n-1}(\mathbf{U}_{n,i}) \setminus \mathbf{h}_{n-1}(\mathbf{V}_{n,i})$$

have moduli uniformly bounded from above and away from zero independent of n and i.

The homotopy $g_t \colon \operatorname{Dom}(\mathbf{h}_{n-1}) \to \mathbb{C}$, for $t \in [0,1]$, is defined as

$$\begin{cases} \mathbf{h}_{n-1}(z) & \text{if } z \notin \bigcup_i \mathbf{V}_{n,i} \\ \chi_i^{-1}\Big((\frac{-t}{3}\sin\frac{(|\chi_i \circ \mathbf{h}_{n-1}(z)|-1)\pi}{4}+1)\cdot \chi_i \circ \mathbf{h}_{n-1}(z)\Big) & \text{if } z \in \mathbf{V}_{n,i}. \end{cases}$$

It is straight to see that $g_0 = \mathbf{h}_{n-1}$ on $\operatorname{Dom} \mathbf{h}_{n-1}$, g_t is a well defined homeomorphism for every fixed t, and g_t depends continuously on t for every fixed z. For every $z \in \partial \mathbf{V}_{n,i}$, at time $t = 1$, we have $g_1(z) = \chi_i^{-1}(\frac{2}{3} \cdot \chi_i \circ \mathbf{h}_{n-1}(z)) \in \partial \widetilde{\mathbf{L}}_{n,i}$. That is, g_1 maps $\partial V_{n,i}$ to $\partial \widetilde{\mathbf{L}}_{n,i}$.

For the other part, we consider q.c. homeomorphisms

$$\Theta_i : \big(\widetilde{\mathbf{U}}_{n,i}, \widetilde{\mathbf{V}}_{n,i}, \widetilde{\mathbf{L}}_{n,i}, \widetilde{\mathbf{J}}_{1,i}\big) \to \big(D_5, D_3, D_2, D_1\big),$$

and define $g_{t+1} : \operatorname{Dom} \mathbf{h}_{n-1} \to \mathbb{C}$, for $t \in [0,1]$, as

$$\begin{cases} g_1(z) & \text{if } z \notin g_1^{-1}(\bigcup_i \widetilde{\mathbf{U}}_{n,i}) \\ \Theta_i^{-1}\Big((\frac{t}{\sqrt{2}}\sin\frac{(|\Theta_i \circ g_1(z)|-1)\pi}{4}+1)\cdot \Theta_i \circ g_1(z)\Big) & \text{if } z \in g_1^{-1}(\widetilde{\mathbf{U}}_{n,i}). \end{cases}$$

The homotopy g_t for $t \in [0,2]$ is the desired adjustment. The map $g_2 : \operatorname{Dom} \mathbf{h}_{n-1} \to \operatorname{Range} \mathbf{h}_{n-1}$, is denoted by $\mathbf{h}'_{n-1}$. □

Let $\Delta_{n-1,0}$ denote the S_{n-1}-preimage of the domain bounded by $E^{\eta(\varepsilon)}$ in the dynamic plane of $\mathbf{f}_{c_{n-1}}$. The domain $\Omega_{n-1,0}$ is defined as

$$\Delta_{n-1,0} \setminus \bigcup_{i=0}^{t_n/t_{n-1}} V_{n,it_{n-1}}.$$

The domains $\Delta_{n-1,i}$ and $\Omega_{n-1,i}$, for $i = 1, 2, \ldots, t_{n-1}$, are defined as the pullback of $\Delta_{n-1,0}$ and $\Omega_{n-1,0}$, respectively, under f^{-i} along the orbit of the critical point.

Consider the map

$$h_{n-1,0} := \widetilde{S}_{n-1}^{-1} \circ \mathbf{h}'_{n-1} \circ S_{n-1} : \Delta_{n-1,0} \to \widetilde{\Delta}_{n-1,0}, \tag{4.5}$$

and then,

$$h_{n-1,i} := \tilde{f}^{-i} \circ h_{n-1,0} \circ f^i : \Delta_{n-1,i} \to \widetilde{\Delta}_{n-1,i}, \tag{4.6}$$

for the appropriate choice of the inverse branch of f^i. As these maps are compositions of two $K_1(\varepsilon)$-q.c. and a $K_3(\varepsilon)$-q.c. (and possibly some conformal maps), they are q.c. with uniformly bounded dilatations. By our adjustment in Lemma 4.7, we know that $h_{n-1,i}$ maps $\Omega_{n-1,i}$ onto $\widetilde{\Omega}_{n-1,i}$.

Finally, the annulus $V_{n-1,0} \setminus W_{n-1,0}$, with modulus bigger than ε/d, encloses $\Omega_{n-1,0}$ and is contained in $V_{n-1,0}$. This proves that $\Omega_{n-1,0}$ is well inside the disk

$D_{n-1,0} := V_{n-1,0}$. Similarly, the appropriate preimage of $V_{n-1,0} \setminus W_{n-1,0}$ under the conformal map f^i introduces a definite annulus around $\Omega_{n-1,i}$ which is contained in $D_{n-1,i} := V_{n-1,i}$. In this case, $D_{n,i}$ is defined as $V_{n,i}$ which contains $J_{n,i}$ well inside itself.

Case $\mathscr{B}$: Here $\mathbf{f}_{c_{n-1}}$ is satellite renormalizable and its second renormalization is of primitive type. Let α_{n-1} denote the dividing fixed point of $\mathbf{f}_{c_{n-1}}$, and $\alpha_n \in \mathbf{J}_{1,0} := J(\mathcal{R}\mathbf{f}_{c_{n-1}})$ denote the dividing fixed point of $\mathcal{R}\mathbf{f}_{c_{n-1}}$. By definition, the little Julia sets $\mathbf{J}_{1,i}$, $i = 1, 2, \ldots, t_n/t_{n-1}$, of $\mathbf{f}_{c_{n-1}}$ touch each other at the α_{n-1} fixed point. Here, $J(\mathcal{R}^2\mathbf{f}_{c_{n-1}})$, and its forward images under iterates of $\mathbf{f}_{c_{n-1}}$ can be arbitrarily close to α_{n-1} (which is a non-dividing fixed point of $\mathcal{R}\mathbf{f}_{c_{n-1}}$). Our idea is to skip the satellite level and start with the primitive one. This essentially imposes the $\mathcal{SL}$ condition on us.

Consider an equipotential $E^{\eta(\varepsilon)}$ contained in $S_{n-1}(W_{n-1,0})$, the external rays landing at α_{n-1}, and the external rays landing at the $\mathbf{f}_{c_{n-1}}$-orbit of α_n (see Figure 6). Let us denote $\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}$ by g, for the simplicity of the notation, throughout this case.

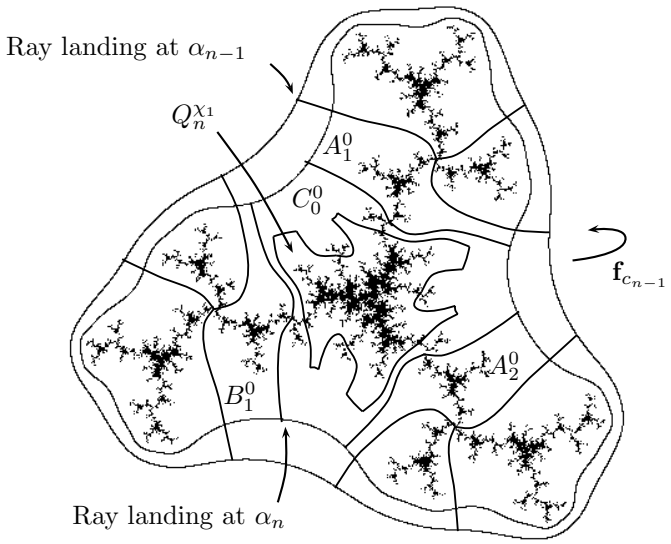

FIGURE 6. The figure shows an infinitely renormalizable Julia set. The first renormalization is of satellite type and the second one is of primitive type. The puzzle piece $Q_n^{\chi_1}$ at the center is the first puzzle piece in the favorite nest.

Let Y_0^1, as before, denote the critical puzzle piece of level one bounded by $E^{\eta(\varepsilon)}$, the external rays landing at α_{n-1}, and the external rays landing at the points in $\mathbf{f}_{c_{n-1}}^{-1}(\alpha_{n-1})$. The external rays landing at α_n and their g-preimage, cut the puzzle piece Y_0^1 into finitely many pieces. Let us denote the one containing the critical

point by C_0^0, the non-critical ones with α_n on their boundary by B_i^0, and the rest of them by A_j^0 (these ones have an $\omega\alpha_n$ on their boundary, with $\omega^d = 1$ but $\omega \neq 1$).

The g-preimage of Y_0^1 along the post-critical set is contained in itself. As all the processes of making the modified principal nest and the pseudo-conjugacy in Theorem 3.2 are based on pullback arguments, the same ideas, which is explained briefly below, work here as well. The only difference is that we do not have any equipotential for the second renormalization of $\mathbf{f}_{c_{n-1}}$. However, certain external rays and part of $E^{\eta(\varepsilon)}$ can play the role of an equipotential for $\mathcal{R}^2(\mathbf{f}_{c_{n-1}})$.

By definition of satellite and primitive renormalizability, $g^n(0)$ belongs to Y_0^1, for $n \geq 0$, and there is a smallest positive integer t with $g^t(0) \in A_1^0$ (by rearranging the indices if required). Pulling A_1^0 back under g^t, along $0, g(0), \dots, g^t(0)$, we obtain a puzzle piece $Q_n^{\chi_1} \ni 0$, such that $C_0^0 \setminus Q_n^{\chi_1}$ is a non-degenerate annulus. That is because C_0^0 is bounded by the external rays landing at α_n and their g-preimage. Therefore, if $Q_n^{\chi_1}$ intersects ∂C_0^0 at some point on the rays, orbit of this intersection under g^k, for $k \geq 1$, stays on the rays landing at α_n. This implies that image of $Q_n^{\chi_1}$ can never be A_1^0. Also, they do not intersect at equipotentials with different levels.

Now, let $m > t$ be the smallest positive integer with $g^m(0) \in Q_n^{\chi_1}$. Pulling $Q_n^{\chi_1}$ back under g^m, along the critical orbit, we obtain $P_n^{\chi_1}$. The map g^m is a unicritical degree d branched covering from $P_n^{\chi_1}$ onto $Q_n^{\chi_1}$. This introduces the first two pieces in the favorite nest. The rest of the process to form the whole favorite nest is the same as in Section 3.2.

Consider the maps $\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}} : Y_0^1 \to \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}(Y_0^1)$, and the corresponding tilde one. One applies Theorem 3.2 to these maps, using the favorite nests introduced in the above paragraph, to obtain a q.c. pseudo-conjugacy

$$\mathbf{h}_{n-1} : \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}(Y_0^1) \to \mathbf{f}_{\tilde{c}_{n-1}}^{t_n/t_{n-1}}(\widetilde{Y}_0^1),$$

up to level of $Q_n^{\chi_n}$. The equipotential $E^{\eta(\varepsilon)}$, the external rays landing at α_{n-1}, and the external rays landing at the $\mathbf{f}_{c_{n-1}}$-orbit of α_n depend holomorphically on the parameter within the secondary wake $W(\eta)$ containing the parameter c_{n-1}. The hyperbolic distance between c_{n-1} and $\tilde{c}_{n-1}$ within one of the finitely many secondary wakes $W(\eta)$ is uniformly bounded in terms of the combinatorial class $\mathcal{SL}$. Also, $\operatorname{mod}(Q_n^{\chi_1} \setminus P_n^{\chi_1})$ is bounded away from zero for the parameters in these secondary limbs. Thus, by Proposition 3.3, $\operatorname{Dil}(\mathbf{h}_{n-1})$ depends only on the *a priori* bounds ε and the combinatorial class $\mathcal{SL}$.

As $\mathbf{f}_{c_{n-1}}^{j} : \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}-j}(Y_0^1) \to \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}(Y_0^1)$, for $j = 1, 2, \dots, t_n/t_{n-1} - 1$, is univalent, we can lift $\mathbf{h}_{n-1}$ onto other puzzle pieces as

$$\mathbf{h}_{n-1} := \mathbf{f}_{\tilde{c}_{n-1}}^{-j} \circ \mathbf{h}_{n-1} \circ \mathbf{f}_{c_{n-1}}^{j} : \mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}-j}(Y_0^1) \to \mathbf{f}_{\tilde{c}_{n-1}}^{t_n/t_{n-1}-j}(\widetilde{Y}_0^1)$$

for these values of j. Since all these maps match the Böttcher marking, they glue together to build a q.c. homeomorphism from a neighborhood of $J(\mathbf{f}_{c_{n-1}})$ to a neighborhood of $J(\mathbf{f}_{\tilde{c}_{n-1}})$. Then, one extends this map, as the identity in the Böttcher coordinates, to a q.c. homeomorphism from the domain bounded by $E^{\eta(\varepsilon)}$ to the domain bounded by $\widetilde{E}^{\eta(\varepsilon)}$.

Finally, by Proposition 4.6 and Lemma 4.7, we adjust $\mathbf{h}_{n-1}$ to obtain a q.c. homeomorphism $\mathbf{h}'_{n-1}$ that satisfies

$$\mathbf{h}'_{n-1}(\mathbf{f}_{c_{n-1}}^{i}(J(\mathcal{R}^2(\mathbf{f}_{c_{n-1}})))) = \mathbf{f}_{\tilde{c}_{n-1}}^{i}(J(\mathcal{R}^2(\mathbf{f}_{\tilde{c}_{n-1}}))),$$

and

$$\mathbf{h}'_{n-1}(S_{n-1}(V_{n+1,it_{n-1}})) = \widetilde{S}_{n-1}(\widetilde{V}_{n+1,it_{n-1}}),$$

for $i = 0, 1, 2, \ldots, t_{n+1}/t_{n-1} - 1$.

Now, $\Delta_{n-1,0}$ is defined as S_{n-1}-pullback of the domain bounded by $E^{\eta(\varepsilon)}$. The domain $\Omega_{n-1,0}$ is

$$\Delta_{n-1,0} \setminus \bigcup_{i=0}^{t_{n+1}/t_{n-1}-1} V_{n+1,it_{n-1}}.$$

The regions $\Delta_{n-1,i}$ and $\Omega_{n-1,i}$, for $i = 1, 2, 3, \ldots, t_{n-1}$, are the appropriate pullbacks of $\Delta_{n-1,0}$ and $\Omega_{n-1,0}$ under f^i, respectively. Like previous case, $h_{n-1,i}$ is defined as in Equation 4.5 or 4.6 and satisfies

$$h_{n-1,i}(\Omega_{n-1,i}) = \widetilde{\Omega}_{n-1,i}, \text{ for } i = 1, 2, \ldots, t_{n-1}.$$

For the same reason as in Case $\mathscr{A}$, $\Omega_{n-1,i}$ is well inside $D_{n-1,i} := V_{n-1,i}$, and $J_{n+1,i}$ is well inside $D_{n+1,i} := V_{n+1,i}$.

Case $\mathscr{C}$: Here, $J(\mathcal{R}(\mathbf{f}_{c_{n-1}}))$ is denoted by $\mathbf{J}_{1,0}$, and $\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}-i}(\mathbf{J}_{1,0})$, for $i = 1, 2, \ldots, \frac{t_n}{t_{n-1}} - 1$, is denoted by $\mathbf{J}_{1,i}$. These little Julia sets touch each other at the dividing fixed point α_{n-1} of $\mathbf{f}_{c_{n-1}}$. Note that α_{n-1} is one of the non-dividing fixed points of $\mathcal{R}\mathbf{f}_{c_{n-1}}$. The union of these little Julia sets is called the *Julia bouquet* and is denoted by $\mathbf{B}_{1,0}$.

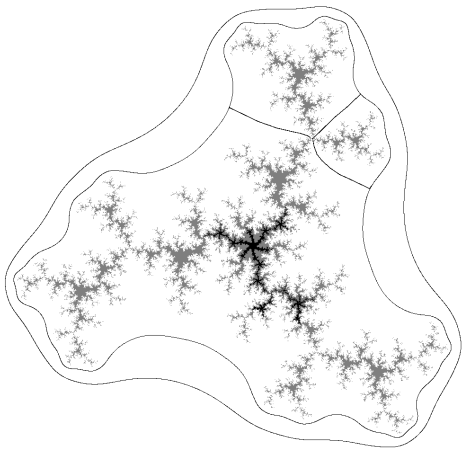

FIGURE 7. A twice satellite renormalizable Julia set drawn in grey. The dark part is the Julia Bouquet $\mathbf{B}_{2,0}$.

Similarly, $J(\mathcal{R}^2(\mathbf{f}_{c_{n-1}}))$, and its forward iterates under $\mathbf{f}_{c_{n-1}}$ are denoted by $\mathbf{J}_{2,i}$, for $i = 0, 1, \ldots, (t_{n+1}/t_{n-1}) - 1$,(i.e. $\mathbf{f}_{c_{n-1}}^i(\mathbf{J}_{2,i}) = \mathbf{J}_{2,0}$). The connected

components of the union of these little Julia sets are called the *bouquets* $\mathbf{B}_{2,j}$, $j = 0, 1, \dots, t_n/t_{n-1} - 1$, (here, $\mathbf{f}_{c_{n-1}}^j(\mathbf{B}_{2,j}) = \mathbf{B}_{2,0}$). In other words,

$$\mathbf{B}_{2,j} := \bigcup_{k=0}^{\frac{t_{n+1}}{t_{n-1}}-1} \mathbf{J}_{2,k\frac{t_n}{t_{n-1}}+j}.$$

Each $\mathbf{B}_{2,j}$ consists of t_{n+1}/t_n little Julia sets touching each other at one of their non-dividing fixed points. As usual, $\mathbf{B}_{2,0}$ denotes the bouquet containing the critical point. See Figure 7.

By an equipotential $E^{\eta(\varepsilon)} \subset S_{n-1}(W_{n-1,0})$, and the external rays landing at α_{n-1}, we form the puzzle pieces of level zero. Recall that Y_0^0 denotes the critical puzzle piece of level zero. The following lemma shows that the bouquets are well apart from each other. See Lemma 10.5 in [Lyu97] for a similar statement.

Lemma 4.8. *For the twice satellite renormalizable parameters in a finite number of truncated secondary limbs,* $\mathrm{mod}\,(Y_0^0 \setminus \mathbf{B}_{2,0})$ *is uniformly bounded from above and away from zero.*

Let X and Y be non-empty compact subsets of $\mathbb{C}$. The *Hausdorff distance* between X and Y is defined as

$$d_H(X, Y) := \inf\{\varepsilon \in \mathbb{R} : Y \subset B_\varepsilon(X), \text{ and } X \subset B_\varepsilon(Y)\},$$

where $B_\varepsilon(X)$ denotes the ε neighborhood of X in the Euclidean metric. The space of all non-empty compact subsets of $\mathbb{C}$ endowed with this metric is a complete metric space.

We say that a family of simply connected domains U_λ, parametrized on a topological disk, depends continuously on λ, if there exist choices of uniformizations $\psi_\lambda : D_1 \to U_\lambda$ continuous in both variables. We say that a family of polynomial-like mappings $(P_\lambda : V_\lambda \to U_\lambda, \lambda \in \Lambda)$, parametrized on a topological disk Λ, depends continuously on λ, if

- U_λ is a continuous family in $\mathbb{C}$, and
- for every fixed $z \in \mathbb{C}$, the map $P_\lambda(z)$, restricted to the values of λ it is defined, depends continuously on λ.

Proposition 4.9. *Let* $(P_\lambda : V_\lambda \to U_\lambda, \lambda \in \Lambda)$ *be a continuous family of polynomial-like mappings with connected filled Julia sets* K_λ*. Then for every* $\lambda \in \Lambda$ *and every* $\varepsilon > 0$*, there exists* $\eta > 0$ *such that if* $|\lambda' - \lambda| < \eta$*, then* $K_{\lambda'} \subseteq B_\varepsilon(K_\lambda)$*.*

Proof. Assume that $z \notin B_\varepsilon(K_\lambda)$. If $z \notin V_\lambda$, then by continuous dependence of V_λ on λ, $z \notin K_{\lambda'}$ for λ' sufficiently close to λ. If $z \in V_\lambda$, then there exists a positive integer l with $P_\lambda^l(z) \in U_\lambda \setminus V_\lambda$. As $P_{\lambda'} : V_{\lambda'} \to U_{\lambda'}$ converges to $P_\lambda : V_\lambda \to U_\lambda$, when $\lambda' \to \lambda$, at least one of $P_{\lambda'}^l(z)$ or $P_{\lambda'}^{l+1}(z)$ belongs to $U_{\lambda'} \setminus V_{\lambda'}$, for λ' sufficiently close to λ. Therefore, $z \notin K_{\lambda'}$. □

Proof of Lemma 4.8. Let c_{n-1} be a twice satellite renormalizable parameter in a given truncated secondary limb. Consider the external rays landing at the dividing fixed point α_n of $\mathcal{R}\mathbf{f}_{c_{n-1}}$ and their preimages under $\mathcal{R}\mathbf{f}_{c_{n-1}}$. Let X_0^0 denote the critical puzzle piece obtained from these rays. As $\mathbf{f}_{c_{n-1}}$ is twice satellite renormalizable, $\mathcal{R}^2\mathbf{f}_{c_{n-1}} : X_0^0 \to \mathbb{C}$ is a proper branched covering over its image. One can consider a continuous thickening of X_0^0, mentioned in Section 3.2, to form a continuous family of polynomial-like mappings parametrized over this truncated limb.

The little filled Julia set of this map is denoted by $\mathbf{K}_{2,0}$, and the filled bouquet $\mathbf{B}_{2,0}$ is the connected component of

$$\bigcup_{i=0}^{t_{n+1}/t_{n-1}-1} \mathbf{f}_{c_{n-1}}(\mathbf{K}_{2,0})$$

containing the critical point.

For a twice satellite renormalizable c_{n-1} in a finite number of truncated secondary limbs, the filled bouquet $\mathbf{B}_{2,0}$ is defined an is a union of a finite number of little filled Julia sets. Also $\mathbf{B}_{2,0}$ is contained well inside the interior of Y_0^0 for c_{n-1} in the closure of these parameter values. Therefore, by the above proposition, $\mathrm{mod}\,(Y_0^0 \setminus \mathbf{B}_{2,j})$ is uniformly bounded away from zero.

To see that these moduli are uniformly bounded from above, one needs only to observe that α_n and 0 belong to $\mathbf{B}_{2,0}$ and are distinct for these parameters. □

By Lemma 4.8, there are simply connected domains $\mathbf{L}'_n \subseteq \mathbf{L}_n$ and $\widetilde{\mathbf{L}}'_n \subseteq \widetilde{\mathbf{L}}_n$, with smooth boundaries, such that the moduli of the annuli

$$\begin{array}{lll} Y_0^0 \setminus \mathbf{L}_n, & \mathbf{L}_n \setminus \mathbf{L}'_n, & \mathbf{L}'_n \setminus \mathbf{B}_{2,0} \\ \widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_n, & \widetilde{\mathbf{L}}_n \setminus \widetilde{\mathbf{L}}'_n, & \widetilde{\mathbf{L}}'_n \setminus \widetilde{\mathbf{B}}_{2,0} \end{array} \tag{4.7}$$

are uniformly bounded from above and away from zero by some constants depending only on the combinatorial class $\mathcal{SL}$. It follows that the ratios

$$\frac{\mathrm{mod}\,(Y_0^0 \setminus \mathbf{L}_n)}{\mathrm{mod}\,(\widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_n)}, \quad \frac{\mathrm{mod}\,(\mathbf{L}_n \setminus \mathbf{L}'_n)}{\mathrm{mod}\,(\widetilde{\mathbf{L}}_n \setminus \widetilde{\mathbf{L}}'_n)}, \quad \frac{\mathrm{mod}\,(\mathbf{L}'_n \setminus \mathbf{B}_{2,0})}{\mathrm{mod}\,(\widetilde{\mathbf{L}}'_n \setminus \widetilde{\mathbf{B}}_{2,0})}$$

are also uniformly bounded from above and away from zero independent of n.

Above data implies that there exists a q.c. homeomorphism

$$\mathbf{h}_{n-1} : Y_0^0 \setminus \mathbf{L}_n \to \widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_n,$$

with uniformly bounded dilatation, which matches the Böttcher marking on ∂Y_0^0 (See Lemma 4.12). Then, we lift $\mathbf{h}_{n-1}$ via $\mathbf{f}_{c_{n-1}}^{-i}$ and $\mathbf{f}_{\tilde{c}_{n-1}}^{-i}$ to extend $\mathbf{h}_{n-1}$ to q.c. homeomorphisms

$$\mathbf{h}_{n-1} : \mathbf{f}_{c_{n-1}}^{-i}(Y_0^0 \setminus \mathbf{L}_n) \to \mathbf{f}_{\tilde{c}_{n-1}}^{-i}(\widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_n), \text{ for } i = 1, 2, \ldots, \frac{t_n}{t_{n-1}} - 1.$$

The domain of each map above is a puzzle piece Y_j^0 (with $j = t_n/t_{n-1} - i$) cut off by E^{η/d^i} and the appropriate component of $\mathbf{f}_{c_{n-1}}^{-i}(\partial \mathbf{L}_n)$. As all these maps match the Böttcher marking on ∂Y_j^0, they can be glued together. Finally, one extends this map onto unbounded components of $\mathbb{C} \setminus \mathrm{Dom}\ \mathbf{h}_{n-1}$ as the identity in the Böttcher coordinates. We denote this extended map with the same notation $\mathbf{h}_{n-1}$. As it is lifted under and extended by holomorphic maps, there is a uniform bound on its dilatation.

Let $\Delta_{n-1,0}$ be the S_{n-1}-preimage of the domain inside $E^{\eta(\varepsilon)}$, and $L_{n,i}$, for $i = 1, 2, \ldots, \frac{t_n}{t_{n-1}} - 1$, be the component of $S_{n-1}^{-1}(\mathbf{f}_{c_{n-1}}^{-i}(\mathbf{L}_n))$ containing $J_{n-1,i} \cap \mathcal{PC}(f)$. Then, define the multiply connected regions

$$\Omega_{n-1,0} := \Delta_{n-1,0} \setminus \bigcup_{i=0}^{t_n/t_{n-1}-1} L_{n,it_{n-1}}.$$

Like before, $\Delta_{n-1,i}$ and $\Omega_{n-1,i}$ are defined as the appropriate f^i-preimage of $\Delta_{n-1,0}$ and $\Omega_{n-1,0}$ intersecting $J_{n-1,i}$, respectively.

We have

$$h_{n-1,0} := \widetilde{S}_{n-1}^{-1} \circ \mathbf{h}_{n-1} \circ S_{n-1} : \Delta_{n-1,0} \to \widetilde{\Delta}_{n-1,0},$$
$$\text{with } h_{n-1,0}(\Omega_{n-1,0}) = \widetilde{\Omega}_{n-1,0}.$$

Also,

$$h_{n-1,i} := \tilde{f}^{-i} \circ h_{n-1,0} \circ f^i : \Delta_{n-1,i} \to \widetilde{\Delta}_{n-1,i},$$
$$\text{with } h_{n-1,i}(\Omega_{n-1,i}) = \widetilde{\Omega}_{n-1,i}.$$

As $E^{\eta(\varepsilon)}$ is contained in $S_{n-1}(W_{n-1,0})$, $\Omega_{n-1,0}$ is contained in $W_{n-1,0}$. Therefore, $\Omega_{n-1,0}$ is well inside $D_{n-1,0} := V_{n-1,0}$. The conformal invariance of modulus implies that the other domains $\Omega_{n-1,i}$ are well inside $D_{n-1,i} := V_{n-1,i}$ as well. This completes the construction in Case $\mathscr{C}$.

To fit together the multiply connected domains $\Omega_{n(k),i}$ and the q.c. homeomorphisms $h_{n(k),i} : \Omega_{n(k),i} \to \widetilde{\Omega}_{n(k),i}$, we follow the string of cases introduced before Lemma 4.5. In Cases $\mathscr{A}$ and $\mathscr{B}$, we have adjusted $\mathbf{h}_{n-1}$, in Lemma 4.7, such that it sends $\partial V_{n,0}$ to $\partial \widetilde{V}_{n,0}$. Therefore, if any of the three cases follows a Case $\mathscr{A}$ or $\mathscr{B}$, we consider $\mathcal{R}^n f : W_{n,0} \to V_{n,0}$ and straighten it with these choices of domains (instead of considering $\mathcal{R}^n f : V_{n,0} \to U_{n,0}$). If the construction on some level n follows a Case $\mathscr{C}$ (level $n-1$ belongs to $\mathscr{C}$), the set $\Delta_{n,0}$ introduced on level n is not contained in the hole $D_{n,0} = L_{n,0}$ obtained on level $n-1$ in Case $\mathscr{C}$. Note that here the n-th renormalization is of satellite type. The following paragraph explains the adjustment needed here.

As the annulus $\mathbf{\Delta}_{n,0} \setminus J(\mathbf{f}_{c_n})$ has definite modulus in terms of ε, the annulus $\mathbf{\Delta}_{n,0} \setminus \mathbf{B}_{1,0}(\supset \mathbf{\Delta}_{n,0} \setminus J(\mathbf{f}_{c_n}))$, where $\mathbf{B}_{1,0} := \cup_{i=0}^{\infty} \mathbf{f}_{c_n}^i(J(\mathcal{R}\mathbf{f}_{c_n}))$, also has definite modulus in terms of ε. By quasi-invariance of modulus, the annulus obtained from

$$S_n(L'_{n,0} \cap \text{Dom } S_n) \setminus \mathbf{B}_{1,0}$$

must have definite modulus in terms of ε. Now, let $\mathbf{E}_n$ be a topological disk contained in

$$S_n(L'_{n,0} \cap \text{Dom } S_n) \cap \mathbf{\Delta}_{n,0}$$

with $\text{mod}\,(\mathbf{E}_n \setminus \mathbf{B}_{1,0})$ bigger than a constant in terms of ε (similarly define the corresponding tilde one). By a similar argument as in Lemma 4.7 we can adjust $\mathbf{h}_n$ through a homotopy to obtain a map $\mathbf{h}'_n : \mathbf{E}_n \to \widetilde{\mathbf{E}}_n$. Now, in this situation $\mathbf{\Delta}_{n,0}$ is replaced by $\mathbf{E}_n$, $\Delta_{n,0}$ by $S_n^{-1}(\mathbf{E}_n)$, and h_n by $\widetilde{S}_n^{-1} \circ \mathbf{h}'_n \circ S_n$. The annulus $L_{n,0} \setminus L'_{n,0}$ provides the definite space around $\Omega_{n,0}$ in $L_{n,0}$.

In the following two sections, we will denote the holes of $\Omega_{n(k),i}$ by $\mathbb{V}_{n(k)+1,j}$. That is, $\mathbb{V}_{n(k)+1,j}$ is $V_{n(k)+1,j}$, if step $n(k)$ belongs to Case $\mathscr{A}$ or $\mathscr{B}$, and $\mathbb{V}_{n(k)+1,j}$ is $S_{n(k)}^{-1}(\mathbf{L}_{n(k),j})$ if step $n(k)$ belongs to Case $\mathscr{C}$.

4.4. **The gluing maps $g_{n(k),i}$.** In this section we build q.c. homeomorphisms

$$g_{n(k),i} : \mathbb{V}_{n(k),i} \setminus \Delta_{n(k),i} \to \widetilde{\mathbb{V}}_{n(k),i} \setminus \widetilde{\Delta}_{n(k),i},$$

with uniformly bounded dilatations, needed to glue the maps $h_{n(k),i}$ together. Every $g_{n(k),i}$ must be identical with $h_{n(k-1),i}$ on $\partial \mathbb{V}_{n(k),i}$, and with $h_{n(k),i}$ on $\partial \Delta_{n(k),i}$ (which is the outer boundary of $\Omega_{n(k),i}$). Then, gluing all these maps $g_{n(k),i}$ and $h_{n(k),i}$ together produces a q.c. homeomorphism denoted by H. In what follows,

for the simplicity of the notation, we use index n instead of $n(k)$, and assume that n runs over the subsequence $\langle n(k)\rangle_{k=1}^{\infty}$. So, by "for all n" we mean "for all $n(k)$".

Like previous steps, first we build $g_{n,0}$, for all n, and then lift them via f^i and $\tilde{f}^i$ to obtain $g_{n,i}$, for $i = 1, 2, \dots, t_n$. Definitions of the maps $h_{n,i}$ as well as the domains $\mathbb{V}_{n,i}$ and $\Omega_{n,i}$ imply that these maps also glue together on the boundaries of their domains of definition. Again, we drop the second subscript if it is zero, i.e., g_n denotes the map $g_{n,0}$.

To build a q.c. homeomorphism from an annulus to an annulus with given boundary conditions, there are $\mathbb{Z}$ possible choices for the number of the "twists" one may make. To have a uniform bound on the dilatation of such a map, not only must the two annuli have comparable moduli uniformly bounded away from zero but also the number of twists must be uniformly bounded as well. Note that the homotopy class of the final map H depends on the choices of these twists.

In this section, we show that the annuli $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$ have comparable moduli. In the next section we prescribe the correct number of twists needed to obtain a Thurston conjugacy.

Lemma 4.10. *There exists a constant M' depending only on ε such that for every $n \geq 1$,*

$$\frac{1}{M'} \leq \frac{\operatorname{mod}(\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0})}{\operatorname{mod}(\mathbb{V}_{n,0} \setminus \Delta_{n,0})} \leq M'.$$

Proof. If level n follows one of Cases $\mathscr{A}$ or $\mathscr{B}$, then

$$\operatorname{mod}(\mathbb{V}_{n,0} \setminus \Delta_{n,0}) \leq \operatorname{mod}(V_{n,0} \setminus J_{n,0}) \leq \eta,$$

by (4.2). If level n follows a Case $\mathscr{C}$ then

$$\operatorname{mod}(\mathbb{V}_{n,0} \setminus \Delta_{n,0}) \leq \operatorname{mod}\, S_n^{-1}(\mathbf{L}_{n,0} \setminus \mathbf{B}_{1,0}) \leq M'',$$

for some constant M'' by the proof of Lemma 4.8 (Here $\mathbf{B}_{1,0}$ is the unique bouquet of $\mathbf{f}_{c_n}$).

Similarly, $\operatorname{mod}(\mathbb{V}_{n,0} \setminus \Delta_{n,0})$ is bigger than ε, or some constant depending on ε, depending on whether level n follows a Case $\mathscr{A}$, $\mathscr{B}$, or $\mathscr{C}$. Hence, $\operatorname{mod}(\mathbb{V}_{n,0} \setminus \Delta_{n,0})$ and $\operatorname{mod}(\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0})$ are pinched between two constants depending only on ε. This implies the lemma. □

Let $A(r)$ denote the round annulus $D_r \setminus D_1$, for $r > 1$, and assume $\gamma : [0,1] \to A(r)$ is a curve parametrized in the polar coordinate as $\gamma(t) = (r(t), \theta(t))$, for $t \in [0,1]$, with $r(t)$ and $\theta(t)$ continuous functions from $[0,1]$ to $\mathbb{R}$. The *wrapping number* of γ, denoted by $\omega(\gamma)$, is defined as $\theta(1) - \theta(0)$. Given a curve $\gamma : [0,1] \to U$, where U is an annulus, with $\gamma(0)$ on the inner boundary of U (corresponding to the bounded component of $\mathbb{C} \setminus U$) and $\gamma(1)$ on the outer boundary of U (corresponding to the unbounded one), we define the wrapping number of γ in U as $\omega(\gamma) := \omega(\phi \circ \gamma)$, where ϕ is a uniformization of U by a round annulus. Note that $\omega(\gamma)$ is invariant under the automorphism group of U. So, it is independent of the choice of the uniformization. In addition, just like winding number, it is constant over the homotopy class of all curves with the same boundary points.

Proposition 4.11. *Given fixed constants $K \geq 1$ and $r > 1$, there exists a constant N such that for every K-q.c. homeomorphism $\psi : A(r) \to A(r')$, the wrapping number of the curve $\psi(t), t \in [1, r]$, belongs to the interval $[-N, N]$.*

Proof. This follows from the compactness of the class of K-q.c. homeomorphisms from $A(r)$ to some $A(r')$. □

In the following lemma let θ be a branch of the argument defined on $\mathbb{C}$ minus an straight ray from 0 to infinity.

Lemma 4.12. *Fix round annuli $A(r)$, $A(r')$, positive constants δ, K_1 and K_2, as well as an integer k with*

$$\operatorname{mod} A(r')/K_1 \leq \operatorname{mod} A(r) \leq K_1 \operatorname{mod} A(r'), \text{ and } \operatorname{mod} A(r) \geq \delta.$$

There exists a constant K depending only on K_1, K_2, k, and δ, such that if homeomorphisms

$$h_1 : \partial D_r \to \partial D_{r'}, \text{ and } h_2 : \partial D_1 \to \partial D_1$$

have K_2-q.c. extensions to some neighborhoods of these circles, then there exists a K-q.c. homeomorphism $h : A(r) \to A(r')$ with:

- *for every $z \in \partial D_r$ we have $h(z) = h_1(z)$, and for every $z \in \partial D_1$ we have $h(z) = h_2(z)$;*
- *The curve $h(t)$, for $t \in [1, r]$, has wrapping number*

$$\theta(h_1(r)) - \theta(h_2(1)) + 2k\pi.$$

If $h_1(r)$ or $h_2(1)$ does not belong to the domain of θ, one may compose h_1 and h_2 with a small rotation. By adding a $+1$ or -1 to k, the statement still holds independent of the choice of this rotation. One proves this lemma by explicitly building such homeomorphisms for every k. Further details are given in the Appendix.

A homeomorphism h as above is called a *gluing* of h_1 and h_2 with k twists. If the number of the twists is not concerned, we say that h is a gluing of h_1 and h_2.

Applying the above lemma to the uniformizations of $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$, with the induced maps from $h_{n-1,0}$ and $h_{n,0}$ on the boundaries, and an integer k_n which will be determined later, gives the K'-q.c. homeomorphisms g_n. In the next section we prescribe some special numbers k_n, which are bounded by a constant depending only on ε, in order to make the K-q.c. homeomorphism H homotopic to a topological conjugacy relative $\mathcal{PC}(f)$.

Definite moduli of the annuli $\mathbb{V}_{n,i} \setminus \Delta_{n,i}$ implies that all the nests of the domains $\mathbb{V}_{n,i}$ shrink to points in $\mathcal{PC}(f)$. Therefore, H can be extended to a well defined K-q.c. homeomorphism on $\mathcal{PC}(f)$. See [Str55] for a detailed proof of this statement.

4.5. **Isotopy.** Let $\boldsymbol{\psi}_n$ denote the topological conjugacy between $\mathbf{f}_{c_n}$ and $\mathbf{f}_{\tilde{c}_n}$ obtained from extending the identity in the Böttcher coordinates through $J(\mathbf{f}_{c_n})$. The lift $\psi_{n,0} := \widetilde{S}_n^{-1} \circ \boldsymbol{\psi}_n \circ S_n$ topologically conjugates $\mathcal{R}^n f$ to $\mathcal{R}^n \tilde{f}$ on a neighborhood of $J_{n,0}$. Note that this neighborhood contains $\Omega_{n,0}$. In the dynamic plane of $\mathbf{f}_{c_n}$, let $U(\eta)$ denote the domain enclosed by E^η. Define

$$\psi_{n,i} := f^{-i} \circ \psi_{n,0} \circ f^i : \Delta_{n,i} \to \mathbb{C},$$

where the inverse branch f^{-i} is chosen so that $\psi_{n,i}(\Delta_{n,i})$ covers a neighborhood of $\mathcal{PC}(f) \cap \tilde{J}_{n,i}$.

Lemma 4.13. *Assume that level n belongs to Case $\mathscr{A}$ or $\mathscr{B}$. For every $i = 0, 1, 2, \ldots, t_n - 1$, $h_{n,i} : \Delta_{n,i} \to \widetilde{\Delta}_{n,i}$ is homotopic to $\psi_{n,i} : \Delta_{n,i} \to \tilde{\Delta}_{n,i}$ relative the little Julia sets $J_{n+1,j}$ of level $n+1$ inside $\Delta_{n,i}$.*

Proof. By the definition of $\Delta_{n,i}$, $\mathbb{V}_{n,i}$, and $h_{n,i}$, it is enough to prove the lemma for $i = 0$. For the other ones, one lifts this homotopy via f^i and $\tilde{f}^i$, or defines them in a similar manner.

Recall that $\Delta_{n,0}$, $\psi_{n,0}$, and $h_{n,0}$ are the lifts of $\boldsymbol{\Delta}_{n,0}$, $\boldsymbol{\psi}_n$, and $\mathbf{h}'_{n,0}$ under the straightening maps. First we introduce a homotopy between $\boldsymbol{\psi}_n$ and $\mathbf{h}'_{n,0}$, relative the Julia sets $\mathbf{J}_{1,i}$ of $\mathbf{f}_{c_n}$, on the dynamic planes of $\mathbf{f}_{c_n}$ and $\mathbf{f}_{\tilde{c}_n}$. Then, we lift this homotopy to the dynamic planes of $\mathcal{R}^n f$ and $\mathcal{R}^n \tilde{f}$ by the straightening maps. Recall that in the construction, $\mathbf{h}'_{n,0}$ is an adjustment of $\mathbf{h}_{n,0}$ through a homotopy relative the little Julia sets $\mathbf{J}_{1,i}$. Thus, to prove the lemma, we need only show that $\mathbf{h}_{n,0}$ in homotopic to $\boldsymbol{\psi}_n$ relative the little Julia sets $\mathbf{J}_{1,i}$.

First assume that level n belongs to Case $\mathscr{A}$. The idea of the proof, presented below in detail, is to partition $\boldsymbol{\Delta}_{n,0}$, by means of rays and equipotential arcs, into several topological disks and an annulus such that $\boldsymbol{\psi}_n$ and $\mathbf{h}_{n,0}$ are identical on the boundaries of these domains.

Recall the puzzle piece $Q^{\chi_n}_{n,0}$ (where $Q^{\chi_n}_{n,0} = Y_0^{q_{\chi_n}}$) introduced in this case. The equipotential $\mathbf{f}_{c_n}^{-\chi_n}(E^\eta)$, and the rays bounding $Q^{\chi_n}_{n,i}$ up to $\mathbf{f}_{c_n}^{-\chi_n}(E^\eta)$, for $i = 0, 1, \dots, t_n - 1$, cut the domain $\boldsymbol{\Delta}_{n,0}$ into the annulus $\boldsymbol{\Delta}_{n,0} \setminus \mathbf{f}^{-\chi_n}(U(\eta))$ and several topological disks. The topological disks which do not intersect the little Julia sets $\mathbf{J}_{1,i}$, the puzzle pieces $Q^{\chi_n}_{n,i}$, and the remaining annulus $U(\eta) \setminus f^{-\chi_n}(U(\eta))$ form the appropriate partition. By Theorem 3.2, $\mathbf{h}_{n,0}$ and $\boldsymbol{\psi}_n$ are identical on the boundaries of these domains. Indeed, $\boldsymbol{\psi}_n$ is the identity in the Böttcher coordinates, and the pseudo-conjugacy $h_{n,0}$ obtained in Theorem 3.2 also matches the Böttcher marking. This proves that the two maps are homotopic outside of the puzzle pieces $Q^{\chi_n}_{n,i}$ relative $\cup_i \partial Q^{\chi_n}_{n,i}$.

To define a homotopy inside $Q^{\chi_n}_{n,0}$, recall that we started with a q.c. homeomorphism $g : Q^{\chi_n}_{n,0} \setminus P^{\chi_n}_{n,0} \to \tilde{Q}^{\chi_n}_{n,0} \setminus \tilde{P}^{\chi_n}_{n,0}$, which was homotopic to $\boldsymbol{\psi}_n$ relative $\partial(Q^{\chi_n}_{n,0} \setminus P^{\chi_n}_{n,0})$. Hence, the lift of g from A^k_n to $\tilde{A}^k_n$, for $k = 1, 2, 3, \dots$, considered in Case $\mathscr{A}$, is homotopic to $\boldsymbol{\psi}_n$ relative ∂A^k_n. As the two maps are identical on $J_{n+1,0} \subset Q^{\chi_n}_{n,0}$, they glue together to define a homotopy between g and $\boldsymbol{\psi}_n$ on $Q^{\chi_n}_{n,0}$ relative $\partial Q^{\chi_n}_{n,0} \cup J_{n+1,0}$.

The same argument applies to all other domains $Q^{\chi_n}_{n,i}$ as well.

If level n belongs to Case $\mathscr{B}$, we repeat the above argument on each puzzle piece Y^0_i, for $i = 0, 1, 2, \dots, \frac{t_n}{t_{n-1}} - 1$.

If level n follows a Case $\mathscr{C}$, one needs to restrict the above homotopies to the smaller domains $\mathbf{E}_{n-1}$ (by the argument at the end of Case $\mathscr{C}$). □

Assume that level n belongs to Case $\mathscr{A}$ or $\mathscr{B}$, and it follows a Case $\mathscr{A}$ or $\mathscr{B}$. Consider the uniformizations

$$\phi_1 : A(s) \to (\mathbb{V}_{n,0} \setminus J_{n,0}), \quad \phi_2 : A(r) \to (\Delta_{n,0} \setminus J_{n,0})$$
$$\widetilde{\phi}_1 : A(\tilde{s}) \to (\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{J}_{n,0}), \quad \widetilde{\phi}_2 : A(\tilde{r}) \to (\widetilde{\Delta}_{n,0} \setminus \widetilde{J}_{n,0})$$

for some constants $s > r$ and $\tilde{s} > \tilde{r}$. The q.c. homeomorphisms $h_{n-1,0} : \mathbb{V}_{n,0} \setminus J_{n,0} \to \widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{J}_{n,0}$ and $h_{n,0} : \Delta_{n,0} \setminus J_{n,0} \to \widetilde{\Delta}_{n,0} \setminus \widetilde{J}_{n,0}$ lift via ϕ_i and $\widetilde{\phi}_i$, $i = 1, 2$, to q.c. homeomorphisms $\hat{h}_{n-1,0} : A(s) \to A(\tilde{s})$ and $\hat{h}_{n,0} : A(r) \to A(\tilde{r})$, respectively, with the same dilatations. By composing these uniformizations with some rotations, if necessary, we may assume that the point one is mapped to the point one under $\hat{h}_{n-1,0}$ and $\hat{h}_{n,0}$. Denote the wrapping number of the image of the line segment

$[1,s]$ under $\hat{h}_{n-1,0}$ by $\omega_{1,n}$, and denote the wrapping number of the image of the line segment $[1,r]$ under $\hat{h}_{n,0}$ by $\omega_{2,n}$. By Proposition 4.11, the absolute values of $\omega_{1,n}$ and $\omega_{2,n}$ are bounded by some constant depending only on ε. Define k_n as $\omega_{1,n}-\omega_{2,n}$. Let $g'_n : D_s \setminus D_r \to D_{\tilde{s}} \setminus D_{\tilde{r}}$ be a gluing of $\hat{h}_{n,0} : \partial D_r \to \partial D_{\tilde{r}}$ and $\hat{h}_{n-1,0} : \partial D_s \to \partial D_{\tilde{s}}$ with k_n twists, using Lemma 4.12. With this choice of gluing, the curve $\hat{h}_{n,0}[1,r] \cup g'_n[r,s]$ is homotopic to the curve $\hat{h}_{n-1,0}[1,s]$ inside $D_s \setminus D_1$ relative the boundary points on $\partial(D_s \setminus D_1)$. Therefore, the map obtained from gluing $\hat{h}_{n,0}$ to g'_n is homotopic to $\hat{h}_{n-1,0} : A(s) \to A(\tilde{s})$ relative the boundary conditions. Let g_n denote the lift of g'_n via ϕ_1 and $\tilde{\phi}_1$. This homotopy lifts to a homotopy between $h_{n-1,0}$ and the map obtained from gluing g_n to $h_{n,0}$.

Before we define k_n in other cases, we need the following extension.

Lemma 4.14. *The q.c. homeomorphism $\mathbf{h}'_{n-1}$ introduced in Case $\mathscr{C}$ admits an extension through $\cup_j \mathbf{L}_{n,j}$ satisfying the following properties*

- *For every j, $\mathbf{h}'_{n-1} : \mathbf{L}_{n,j} \setminus \mathbf{B}_{2,j} \to \widetilde{\mathbf{L}}_{n,j} \setminus \widetilde{\mathbf{B}}_{2,j}$ is q.c. with uniformly bounded dilatation depending only on ε;*
- *$\mathbf{h}'_{n-1} = \boldsymbol{\psi}_{n-1}$ on $\cup_j \mathbf{B}_{2,j}$;*
- *$\mathbf{h}'_{n-1}$ is homotopic to $\boldsymbol{\psi}_{n-1}$ relative $\cup_j \mathbf{B}_{2,j}$.*

Proof. First we extend the map through $\mathbf{L}_{n,0}$. Consider the fundamental annuli $S_{n-1}(U_{n,0} \setminus V_{n,0})$ and $\widetilde{S}_{n-1}(\widetilde{U}_{n,0} \setminus \widetilde{V}_{n,0})$ for $\mathcal{R}\mathbf{f}_{c_{n-1}}$ and $\mathcal{R}\mathbf{f}_{\tilde{c}_{n-1}}$. Let

$$g_n : S_{n-1}(U_{n,0} \setminus V_{n,0}) \to \widetilde{S}_{n-1}(\widetilde{U}_{n,0} \setminus \widetilde{V}_{n,0})$$

be a q.c. homeomorphism with $\mathcal{R}\mathbf{f}_{\tilde{c}_{n-1}} \circ g_n = g_n \circ \mathcal{R}\mathbf{f}_{c_{n-1}}$ on $\partial(S_{n-1}(V_{n,0}))$. Lifting g_n onto the preimages of these annuli, via $\mathcal{R}\mathbf{f}_{c_{n-1}}$ and $\mathcal{R}\mathbf{f}_{\tilde{c}_{n-1}}$, we obtain a q.c. homeomorphism, still denoted by g_n, from $S_{n-1}(U_{n,0}) \setminus J(\mathcal{R}^1(\mathbf{f}_{c_{n-1}}))$ to $\tilde{S}_{n-1}(\tilde{U}_{n,0}) \setminus J(\mathcal{R}^1(\mathbf{f}_{\tilde{c}_{n-1}}))$. By Lemma 4.6, g_n (or some rotation of it) can be extended onto $J(\mathcal{R}\mathbf{f}_{c_{n-1}}) = \mathbf{J}_{1,0}$ as $\boldsymbol{\psi}_{n-1}$. Moreover, these two maps are homotopic relative $\mathbf{J}_{1,0}$. By a similar argument as in Lemma 4.7, we adjust g_n, through a homotopy relative $\mathbf{B}_{2,0}$, to a q.c. homeomorphism mapping $\mathbf{L}'_n$ to $\widetilde{\mathbf{L}}'_n$.

Consider the three annuli $Y_0^0 \setminus \mathbf{L}_n$, $\mathbf{L}_n \setminus \mathbf{L}'_n$, and $\mathbf{L}'_n \setminus \mathbf{B}_{2,0}$, as well as the corresponding tilde ones. We have

$$\mathbf{h}'_{n-1} : Y_0^0 \setminus \mathbf{L}_n \to \widetilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_n, \text{ and } \quad g_n : \mathbf{L}'_n \setminus \mathbf{B}_{2,0} \to \widetilde{\mathbf{L}}'_n \setminus \widetilde{\mathbf{B}}_{2,0}.$$

To find a gluing of these two maps on the middle annulus $\mathbf{L}_n \setminus \mathbf{L}'_n$, we use the above argument to find the right number of twists on this annulus. Consider a curve γ connecting a point $a \in \mathbf{B}_{2,0}$ to a point $d \in \partial Y_0^0$ such that it intersects each of $\partial \mathbf{L}'_n$ and $\partial \mathbf{L}_n$ only once at points denoted by b and c, respectively. Let us denote by γ_{ab}, γ_{bc}, and γ_{cd} each segment of this curve cut off by these four points. The wrapping number $\omega(\boldsymbol{\psi}_{n-1}(\gamma))$ is uniformly bounded depending only on $\mathcal{SL}$ condition. That is because $\boldsymbol{\psi}_{n-1}$ depends continuously on $\tilde{c}_{n-1}$, and $\tilde{c}_{n-1}$ belongs to a compact set of parameters. Therefore, by Proposition 4.11, $\omega(\boldsymbol{\psi}_{n-1}(\gamma)) - \omega(\mathbf{h}_{n-1}(\gamma_{cd})) - \omega(g_n(\gamma_{ab}))$ is uniformly bounded depending only on ε and the class $\mathcal{SL}$. If we glue $\mathbf{h}_{n-1}$ to g_n by such a number of twists (Lemma 4.12), the resulting map will be homotopic to $\boldsymbol{\psi}_{n-1}$ relative $\mathbf{B}_{2,0} \cup \partial Y_0^0$.

Similarly, one extends $\mathbf{h}'_{n-1}$ onto the other topological disks $\mathbf{L}_{n,j}$. □

If a Case $\mathscr{C}$ follows a Case $\mathscr{A}$ or a Case $\mathscr{B}$, the number of twists k_n is defined as in the proof of the above lemma. If level $n-1$ belongs to Case $\mathscr{C}$ and level n is any

of the three cases, we define k_n using the uniformizations of the annuli $\mathbb{V}_{n,0}, \setminus B_{n,0}$, $\Delta_{n,0} \setminus B_{n,0}$, $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{B}_{n,0}$, and $\widetilde{\Delta}_{n,0} \setminus \widetilde{B}_{n,0}$ instead of the uniformizations after the proof of Lemma 4.13.

The following elementary lemma, whose proof appears in the Appendix, is used to put together the homotopies on different levels.

Lemma 4.15. *Let U and $\widetilde{U}$ be closed annuli with outer boundaries γ_1 and $\tilde{\gamma}_1$ as well as inner boundaries γ_2 and $\tilde{\gamma}_2$, respectively. Also, let $h_1 : \gamma_1 \to \tilde{\gamma}_1$ be a homeomorphism and $h_2^t : \gamma_2 \to \tilde{\gamma}_2$, for $t \in [0,1]$, be a continuous family of homeomorphisms. Any continuous gluing $G^0 : U \to \widetilde{U}$ of h_1 and h_2^0 extends to a continuous family of gluing $G^t : U \to \widetilde{U}$ of h_1 and h_2^t, for $t \in [0,1]$.*

Proposition 4.16. *The q.c. homeomorphism H, obtained from gluing all the maps $g_{n,i}$ and $h_{n,k}$, is homotopic to the topological conjugacy Ψ between f and $\tilde{f}$ relative $\mathcal{PC}(f)$.*

Proof. Let H_n denote the homeomorphism obtained from gluing the maps $g_{i,j}$ and $h_{k,l}$, with i and k less than or equal to n.

First, we claim that

- the maps H_1 and Ψ belong to the same homotopy class of homeomorphisms from $\mathbb{C} \setminus \cup_i J_{1,i}$ to $\mathbb{C} \setminus \cup_i \widetilde{J}_{1,i}$, and
- for every $n > 1$, H_{n-1} and H_n, belong to the same homotopy class of homeomorphisms from $\mathbb{C} \setminus \cup_i J_{n+1,i}$ to $\mathbb{C} \setminus \cup_i \widetilde{J}_{n+1,i}$.

By definition, H_1 is equal to $h_{1,0}$ which is homotopic to $\psi_{1,0}$ relative $\cup_i J_{1,i}$, by Lemma 4.13 or Lemma 4.14. Therefore, H_1 is homotopic to Ψ, relative $\cup_i J_{1,i}$, by Proposition 4.6.

Recall that H_{n-1} and H_n are identical on the complement of $\cup_j \mathbb{V}_{n,j}$. On $\Delta_{n,i}$, H_{n-1} and H_n are equal to $h_{n-1,i}$ and $h_{n,i}$, respectively.

The domain $\mathbb{V}_{n,0}$ is partitioned into $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\Delta_{n,0}$. On $\Delta_{n,0}$, the maps $h_{n,0}$ and $h_{n-1,0}$ are homotopic to $\psi_{n,0}$ relative $\cup_i J_{n+1,i}$, either by Lemma 4.13 or 4.14. Thus, there exists a homotopy h_n^t, for t in $[0,1]$, which starts with $h_{n,0}$, ends with $h_{n-1,0}$, and maps $\partial\Delta_{n,0}$ to $\partial\widetilde{\Delta}_{n,0}$, for all $t \in [0,1]$. At time zero, consider the map $h_{n,0}$ on the inner boundary of $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$, $h_{n-1,0}$ on the outer boundary of this annulus, and the gluing $G_n^0 := g_{n,0}$ on the annulus. Applying Lemma 4.15 with $h_{n-1,0}$ on the outer boundary and h_n^t on the inner boundary, we obtain a continuous family of gluings $G_n^t : \mathbb{V}_{n,0} \setminus \Delta_{n,0} \to \widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$ between them. The homeomorphism G_n^1 is a gluing of $h_{n-1,0} : \partial\mathbb{V}_{n,0} \to \partial\widetilde{\mathbb{V}}_{n,0}$ and $h_{n-1,0} : \partial\Delta_{n,0} \to \partial\widetilde{\Delta}_{n,0}$. But, G_n^1 must be homotopic to $h_{n-1,0}$ on $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ relative the boundaries. That is because these two maps send a curve joining the two different boundaries to two curves (joining the two boundaries) which are homotopic in the annulus $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$ relative end points. This comes from our choice of the number of twists for the gluing maps. This completes the proof of the claim.

Let $t_0 = 0 < t_1 < t_2 < \cdots$, be an increasing sequence in $[0,1]$ converging to 1. Assume that H^t, for t in $[t_0, t_1]$, denotes the homotopy obtained above between Ψ and H_1 relative the little Julia sets $J_{1,i}$. Also, let H^t, for $t \in [t_n, t_{n+1}]$, $n = 1, 2, \ldots$, denote the homotopy between H_n and H_{n+1} relative the little Julia sets of level $n+2$.

It follows from the construction that for any fixed z, $H^t(z)$ eventually stabilizes and equals to $H(z)$. Indeed, the *a priori* bounds assumption implies that the

diameters of the topological disks $\mathbb{V}_{n,i}$ tend to zero as $n \to \infty$. Therefore, the supremum distance between H^t and H tends to zero as $t \to 1$. We conclude that H^t, for t in $[0,1]$, defines a homotopy between Ψ and H relative $\mathcal{PC}(f)$. Hence, H is a Thurston conjugacy between f and $\tilde{f}$. □

4.6. **Promotion to hybrid conjugacy.**

Proposition 4.17. *Suppose that all infinitely renormalizable unicritical polynomials in a given combinatorial class $\tau = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots\}$ satisfying the $\mathcal{SL}$ condition, enjoy a priori bounds. Then q.c. conjugate maps in this class are hybrid conjugate.*

Proof. Assume that there are polynomials P_1 and P_2 in τ which are q.c. equivalent but not hybrid equivalent. Define the set

$$\begin{aligned}\Omega :=&\{c \in \mathbb{C} : P_c \text{ is q.c. equivalent to } P_1\}\\ =&\{c \in \mathbb{C} : P_c \text{ is q.c. equivalent to } P_2\}.\end{aligned}$$

The plan is to show that Ω is both open and closed in $\mathbb{C}$, which is not possible.

Theorem 4.1 implies that q.c. equivalence is the same as combinatorial equivalence in the class τ. Since every combinatorial class is an intersection of a nest of closed sets (connectedness locus copies), Ω is closed.

Consider a point P in Ω. The polynomial P is not hybrid equivalent to both of P_1 and P_2 by the assumption. Assume that it is not hybrid equivalent to P_1 (for the other case just change P_1 to P_2). Let $\phi_1 : \mathbb{C} \to \mathbb{C}$, be a k-q.c. homeomorphism with $\phi_1 \circ P = P_1 \circ \phi_1$. Replacing $P_1(z)$ by $\omega^{-1}P_1(\omega z)$, for some $(d-1)$-th root of unity ω, if necessary, we may further assume that ϕ_1 is the identity in the Böttcher coordinates. Pulling the standard complex structure μ_0 on $\mathbb{C}$ back under ϕ_1, we obtain a nontrivial P-invariant complex structure μ on $\mathbb{C}$ with dilatation bounded by $\frac{k-1}{k+1}$. Consider the family of complex structures $\mu_\lambda := \lambda \cdot \mu$, for λ in the disk of radius $\frac{k+1}{k-1}$ centered at zero $D(0, \frac{k+1}{k-1}) \subseteq \mathbb{C}$.

By the measurable Riemann mapping Theorem [Ahl06], there exists a unique q.c. mapping $\phi_\lambda : \mathbb{C} \to \mathbb{C}$, for every $\lambda \in D(0, \frac{k+1}{k-1})$, with $\phi_\lambda^* \mu_\lambda = \mu_0$, $\phi_\lambda(0) = 0$, and $\phi_\lambda(z) = z + O(1)$. The map $P_\lambda := \phi_\lambda^{-1} \circ P \circ \phi_\lambda : \mathbb{C} \to \mathbb{C}$, for $\lambda \in D(0, \frac{k+1}{k-1})$, preserves the standard complex structure μ_0. By Weyl's Lemma ([Ahl06], Chapter II, Corollary 2), P_λ is holomorphic. As P_λ is conjugate to P by ϕ_λ, P_λ must be a unicritical polynomial of degree d, and ϕ_λ must map the critical value of P to the critical value of P_λ. At $\lambda = 1$ we obtain P, and at $\lambda = 0$ we obtain P_1. By the analytic dependence of the solution of the measurable Riemann mapping Theorem on the complex structure, the family P_λ, for $\lambda \in D(0, \frac{k+1}{k-1})$, covers a neighborhood of P in Ω. That is because the critical value of P_λ is equal to the ϕ_λ of the critical value of P, and ϕ_λ depends analytically on λ. This shows that P is an interior point of Ω, and, therefore, Ω is open. □

5. Dynamical description of the combinatorics

Here we give a detailed dynamical description of the combinatorial classes mentioned in the Introduction. Let P_c be an infinitely renormalizable unicritical polynomial with renormalizations $f_n := \mathcal{R}^n(P_c)$, $n = 0, 1, 2, \dots$. Each f_n is hybrid conjugate to a polynomial denoted by $\mathbf{f}_{c_n}$. Let $Y_0^1(n)$ denote the critical puzzle piece of level 1 of $\mathbf{f}_{c_n}$. The dynamical meaning of a parameter c satisfying the

decoration condition is that there exists a constant M such that for every $n \geq 0$ there are integers t_n and q_n, both bounded by M, with

- $\mathbf{f}_n^{kq_n}(0) \in Y_0^1(n)$, for every positive integer $k < t_n$, and
- $\mathbf{f}_n^{t_n q_n}(0) \notin Y_0^1(n)$.

In particular, this condition implies that the number of rays landing at the dividing fixed point of $\mathbf{f}_{c_n}$ (denoted by q_n here) is uniformly bounded independent of n.

An infinitely renormalizable parameter is of *bounded type* if the supremum of the relative return times t_{n+1}/t_n, where $\mathcal{R}^n(P_c)$ is an appropriate restriction of $P_c^{t_n}$, is bounded. By definition, the class of maps satisfying the decoration condition contains the infinitely primitively renormalizable parameters of bounded type.

In Section 3.4, we associated a sequence of maximal connectedness locus copies $\tau(f) = \mathcal{M}_d^1, \mathcal{M}_d^2, \dots$ to every infinitely renormalizable unicritical polynomial-like map f. Let $\pi_n(\tau(f)) := \mathcal{M}_d^n$. Define

$$\tau(f,n) := \left\{ c \in \mathcal{M}_d \;\middle|\; \begin{array}{l} P_c(z) = z^d + c \text{ is at least } n \text{ times} \\ \text{renormalizable, and} \\ \pi_i(\tau(f)) = \pi_i(\tau(P_c)), \text{ for } i = 1, 2, \dots, n \end{array} \right\}.$$

Given an infinitely renormalizable map f and a sequence of integers $n_0 = 0 < n_1 < n_2 < \cdots$, we define the sequence:

$$(\tilde{\tau}(f), \langle n_i \rangle) := \langle \tilde{\mathcal{M}}^{n_1}, \tilde{\mathcal{M}}^{n_2}, \dots, \tilde{\mathcal{M}}^{n_k}, \dots \rangle,$$

where

$$\tilde{\mathcal{M}}^{n_k} := \tau(\mathcal{R}^{n_{k-1}} f, n_k - n_{k-1}).$$

Given a sequence of integers $n_0 = 0 < n_1 < n_2 < \cdots$, one can see that there is a one to one correspondence between the two sequences $\tau(f)$ and $(\tilde{\tau}(f), \langle n_i \rangle)$. Thus, one may take the latter one as the definition of the combinatorics of an infinitely renormalizable map.

Consider the main hyperbolic component of $\mathcal{M}_d$. There are infinitely many hyperbolic components of $\mathcal{M}_d$ attached to this component, called primary components. Similarly, there are infinitely many hyperbolic components, secondary ones, attached to these primary components, and so forth. Consider the set of all hyperbolic components obtained this way, i.e., the ones that can be connected to the main hyperbolic component by a chain of hyperbolic components bifurcating one from another. The closure of this set, plus all possible bounded components of its complement, is called the *molecule* $\mathscr{M}_d$.

An infinitely renormalizable map f is said to satisfy the *molecule condition*, if there exists a constant $\eta > 0$ and an increasing sequence of positive integers $n_0 = 0 < n_1 < n_2 < \cdots$, such that for all $i \geq 1$

- $\mathcal{R}^{n_i} f$ is a primitive renormalization of $\mathcal{R}^{n_i - 1} f$, and
- the Euclidean distance between $\tilde{\mathcal{M}}_d^{n_i}$ and $\mathscr{M}_d$ is at least η.

Note that for a map satisfying this condition, there may be infinitely many satellite renormalizable maps in the sequence $\langle \mathcal{R}^n f \rangle$. However, the condition requires that there are infinitely many primitive levels with the corresponding relative connectedness locus copies uniformly away from $\mathscr{M}_d$. By a compactness argument, one can see that the decoration condition is stronger than the molecule condition.

For every $\varepsilon \geq 0$, and every hyperbolic component of $\mathcal{M}_d$, there are at most finitely many limbs attached to this hyperbolic component with diameter bigger

than ε (by the Yoccoz inequality on the size of limbs [Hub93]). This implies that for every $\eta > 0$, all but a finite number of the secondary limbs are contained in the η neighborhood of $\mathscr{M}_d$. This implies that the parameters satisfying the molecule condition also satisfy the $\mathcal{SL}$ condition. Therefore, combining with [KL09b] and [KL08] we obtain the corollary stated in the Introduction.

Appendix

Proof of Proposition 4.6. Consider an external ray R landing at a non-dividing fixed point β_0 of f. As R is invariant under f, and ϕ commutes with f, $\phi(R)$ is also invariant under f. This implies that $\phi(R)$ lands at a non-dividing fixed point β_j of f. Choose ρ_j such that $\rho_j(\phi(R))$ lands at β_0. Let ψ denote the map $\rho_j \circ \phi$, and R' denote the ray $\rho_j(\phi(R))$. For such a rotation ρ_j, ψ also commutes with f, and R' is also invariant under f.

The external ray R cuts the annulus $V_1 \setminus V_2$ into a quadrilateral $I_{0,1}$. The preimage $f^{-1}(I_{0,1})$ consists of d quadrilaterals denoted by $I_{1,1}, I_{1,2}, \dots, I_{1,d}$, ordered clockwise starting with R. Similarly, the $f^{-n}(I_{0,1})$ produces d^n quadrilaterals $I_{n,1}, I_{n,2}, \dots, I_{n,d^n}$ (ordered similarly). In the same way, the external ray R' produces quadrilaterals denoted by $I'_{n,j}$, ordered clockwise starting with R'. We claim that the Euclidean diameter of $I_{n,j}$ (and $I'_{n,j}$) goes to zero as n tends to infinity.

Denote $f^{-i}(V_1)$ by V_{i+1}, and let d_{i+1} denote the hyperbolic metric on the annulus $V_{i+1} \setminus K(f)$. As $I_{n,j} \subseteq V_n$, and $\cap_n V_n = K(f)$, the quadrilaterals $I_{n,j}$ converge to the boundary of $V_1 \setminus K(f)$ as n goes to infinity. To show that the Euclidean diameters of these quadrilaterals go to zero, it is enough to show that their hyperbolic diameters in $(V_1 \setminus K(f), d_1)$ stay bounded from above. Since $f^{n-1} : (V_n \setminus K(f), d_n) \to (V_1 \setminus K(f), d_1)$ is an unbranched covering of degree d^{n-1}, it is a local isometry. As the closure of $f^{n-1}(I_{n,j})$ is a compact subset of $V_1 \setminus K(f)$, we conclude that $I_{n,j}$ has bounded hyperbolic diameter in $(V_n \setminus K(f), d_n)$. Finally, the contraction of the inclusion from $(V_n \setminus K(f), d_n)$ into $(V_1 \setminus K(f), d_1)$ implies that $I_{n,j}$ has bounded hyperbolic diameter in $(V_1 \setminus K(f), d_1)$.

By a similar argument, one can show that the hyperbolic distance between $I_{n,j}$ and $I'_{n,j}$ in $(V_1 \setminus K(f), d_1)$ is uniformly bounded from above.

Since ψ is a conjugacy, it sends $I_{n,j}$ to $I'_{n,j}$. Therefore, as w converges to $K(f)$, w and $\psi(w)$ belong to some $I_{n,j}$ and $I'_{n,j}$, respectively, with larger and larger values of n. Combining with the above argument, we conclude that the Euclidean distance between w and $\psi(w)$ tends to zero as $w \to K(f)$. This implies that ψ extends through $K(f)$ as the identity. □

Proof of Lemma 4.12. Let $\Pi_r := \{z \mid 0 < \mathrm{Im}(z) < \frac{1}{2\pi} \log r\}$, for $r > 1$, denote the covering space of $A(r)$ with the deck transformation group generated by $z \to z+1$. We may assume that $\log r$ and $\log r'$ are at least 6π. Otherwise, one may rescale these strips by affine maps of the form $(x, y) \mapsto (x, ay)$ and continue with the following argument.

The homeomorphisms h_1 and h_2 lift to 1-periodic homeomorphisms

$$\hat{h}_1 : \mathbb{R} + \frac{\mathbf{i}}{2\pi}\log r \to \mathbb{R} + \frac{\mathbf{i}}{2\pi}\log r', \text{ and } \hat{h}_2 : \mathbb{R} \to \mathbb{R},$$
$$\text{with } \hat{h}_2(0) = 0 \text{ , and}$$
$$\hat{h}_1(\frac{\mathbf{i}}{2\pi}\log r) = \theta(h_1(r)) - \theta(h_2(1)) + 2\pi k + \frac{\mathbf{i}}{2\pi}\log r'.$$

As these maps have K_2-q.c. extensions to some neighborhoods of their domains of definition, they are quasi-symmetric with a constant $M(K_2)$ (See Theorem 1 in [Ahl06], Page 40).

To prove the lemma, it is enough to introduce a 1-periodic q.c. mapping $h : \Pi_r \to \Pi_{r'}$, matching $\hat{h}_1$ and $\hat{h}_2$ on the boundaries, and with uniformly bounded dilatation in terms of K_1, K_2, k, and δ.

Define $\phi : \Pi_\infty \to \Pi_\infty$ as $\phi(x, y) := u(x, y) + \mathbf{i}v(x, y)$, where

$$u(x,y) := \frac{1}{2y}\int_{-y}^{+y} \hat{h}_2(x+t)\,dt,$$
$$v(x,y) := \frac{1}{2y}\int_{0}^{y} (\hat{h}_2(x+t) - \hat{h}_2(x-t))\,dt.$$

It has been proved in [Ahl06], Page 42, that ϕ is a q.c. mapping with dilatation depending only on $M(K_2)$. Note that ϕ is 1-periodic in the first variable. It follows from Lemma 3 on Page 41 of [Ahl06] that

$$v(0,1) \le \int_0^1 \hat{h}_2(t)\,dt - \frac{1}{2} \le \frac{M(K_2)}{M(K_2)+1} \le \frac{1}{2} \in [0,3].$$

Also,

$$v_x(x,1) = 0, \text{ and } u_x(x,1) = 1, \text{ for } -\infty < x < \infty.$$

This implies that ϕ maps the horizontal line through $\mathbf{i}$ to a horizontal line in $\Pi_{r'}$ as a translation. Similarly, one extends $\hat{h}_1$ to a q.c. homeomorphism ψ of $\{z \mid \mathrm{Im}(z) \le \frac{1}{2\pi}\log r\}$ such that it maps the horizontal line through $\frac{\mathbf{i}}{2\pi}\log r - 1$ to a horizontal line in $\Pi_{r'}$ as a translation.

Consider the map $H : \{z \mid 1 \le \mathrm{Im}(z) \le \frac{1}{2\pi}\log r - 1\} \to \Pi_{r'}$ defined by

$$(x, y) \mapsto (1 - y)\phi(x, 1) + y\psi(x, \frac{1}{2\pi}\log r - 1).$$

The homeomorphism H is affine with dilatation depending only on k. The homeomorphisms ϕ and ψ are q.c. with dilatations depending only on $M(K_2)$. The homeomorphism

$$\begin{cases} \phi(x,y) & \text{if } 0 \le x \le 1 \\ H(x,y) & \text{if } 1 \le x \le \frac{1}{2\pi}\log r - 1 \\ \psi(x,y) & \text{if } \frac{1}{2\pi}\log r - 1 \le x \le \frac{1}{2\pi}\log r \end{cases}$$

is the desired q.c. mapping, with dilatation depending only on k, K_1, δ, and $M(K_2)$. □

Proof of Lemma 4.15. As this is a purely topological statement, we may assume that U and $\widetilde{U}$ are $D_2 \setminus D_1$ whose universal covering space is $\Pi := \mathbb{R} \times (0, 1)$. Lifting h_1, h_2^t, and G^0 via the projections from the closure of Π to the closures of U and $\widetilde{U}$, we obtain homeomorphisms $\hat{h}_1 : \mathbb{R} \times \{0\} \to \mathbb{R} \times \{0\}$, $\hat{h}_2^t : \mathbb{R} \times \{1\} \to \mathbb{R} \times \{1\}$, and

$\hat{G}^0 : \Pi \to \Pi$, all 1-periodic in the first coordinate. Composing all the above maps with a rotation if necessary, we may assume that $\hat{h}_1(0,0) = (0,0)$. The lift $\hat{G}^0$ is uniquely determined by being an extension of $\hat{h}_1$, and then $\hat{h}_2$ is a unique extension of $\hat{G}^0$. Define the continuous family of 1-periodic homeomorphisms $T^t : \Pi \to \Pi$, for $t \in [0,1]$, as

$$T^t(x,y) := (1-y) \cdot \hat{h}_1(x,0) + y \cdot \hat{h}_2^t(x,1).$$

By defining the 1-periodic homeomorphism $H : \Pi \to \Pi$ as $(\hat{G}^0)^{-1} \circ T^0$, one can see that $T^t \circ H^{-1}$ is a continuous family of gluings of $\hat{h}_1$ and $\hat{h}_2^t$ starting with $\hat{G}^0$. This periodic family projects to a continuous family of interpolations of $\hat{h}_1$ and $\hat{h}_2^t$ on U. □

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