M3S3/M4S3
ASSESSED COURSEWORK 1 : SOLUTIONS

1. We use the result from the handout

$$\sqrt{n}\left(\left(\begin{array}{c} X_{(k_1)} \\ X_{(k_2)} \end{array}\right) - \left(\begin{array}{c} x_{p_1} \\ x_{p_2} \end{array}\right)\right) \xrightarrow{\mathfrak{L}} N\left(0, \left[\begin{array}{cc} \dfrac{p_1(1-p_1)}{\{f_X(x_{p_1})\}^2} & \dfrac{p_1(1-p_2)}{f_X(x_{p_1})\, f_X(x_{p_2})} \\ \\ \dfrac{p_1(1-p_2)}{f_X(x_{p_1})\, f_X(x_{p_2})} & \dfrac{p_2(1-p_2)}{\{f_X(x_{p_2})\}^2} \end{array}\right]\right)$$

which implies, marginally, that, for arbitrary p, and $k = \lceil np \rceil$,

$$\sqrt{n}\left(X_{(k)} - x_p\right) \xrightarrow{\mathfrak{L}} N\left(0, \frac{p(1-p)}{\{f_X(x_p)\}^2}\right)$$

(i) For the **sample median**, $k = \lceil np \rceil$ with $p = 0.5$, we have

$$\sqrt{n}\left(X_{(k)} - x_{0.5}\right) \xrightarrow{\mathfrak{L}} N\left(0, \frac{0.5(1-0.5)}{\{f_X(x_{0.5})\}^2}\right)$$

Now, in the $N(\mu, \sigma^2)$ case, $x_p = \mu$, and $\{f_X(\mu)\}^2 = 1/(2\pi\sigma^2)$, so we have

$$\sqrt{n}\left(X_{(k)} - \mu\right) \xrightarrow{\mathfrak{L}} N\left(0, \frac{0.5(1-0.5)}{1/(2\pi\sigma^2)}\right) \equiv N\left(0, \frac{\pi\sigma^2}{2}\right)$$

and hence

$$X_{(k)} \sim AN\left(\mu, \frac{\pi\sigma^2}{2n}\right).$$

Note that $\pi/2 \approx 1.57 > 1$, so the asymptotic variance is greater for the sample **median** than for the sample **mean**.

[5 MARKS]

(ii) For the **sample interquartile range**, R_{IQ}; $k_1 = \lceil np_1 \rceil$ and $k_2 = \lceil np_2 \rceil$ with $p_1 = 0.25$ and $p_2 = 0.75$. Then from above

$$\sqrt{n}\left(\left(\begin{array}{c} X_{(k_1)} \\ X_{(k_2)} \end{array}\right) - \left(\begin{array}{c} x_{0.25} \\ x_{0.75} \end{array}\right)\right) \xrightarrow{\mathfrak{L}} N\left(0, \left[\begin{array}{cc} \dfrac{0.25(1-0.25)}{\{f_X(x_{0.25})\}^2} & \dfrac{0.25(1-0.75)}{f_X(x_{0.25})\, f_X(x_{0.75})} \\ \\ \dfrac{0.25(1-0.75)}{f_X(x_{0.25})\, f_X(x_{0.75})} & \dfrac{0.75(1-0.75)}{\{f_X(x_{0.25})\}^2} \end{array}\right]\right)$$

Now, by the results given for the standard normal

$$x_{0.25} = \mu - 0.674\sigma \qquad x_{0.75} = \mu + 0.674\sigma$$

and so, by elementary transformation theory

$$\begin{aligned} f_X(x_{0.25}) &= f_X(\mu - 0.674\sigma) = \phi(-0.674)/\sigma = 0.318/\sigma \\ f_X(x_{0.75}) &= f_X(\mu + 0.674\sigma) = \phi(0.674)/\sigma = 0.318/\sigma \end{aligned}$$

Thus, the asymptotic variance-covariance matrix is

$$\Sigma = \frac{\sigma^2}{0.318^2}\left[\begin{array}{cc} 0.25\times 0.75 & 0.25\times 0.25 \\ 0.25\times 0.25 & 0.75\times 0.25 \end{array}\right] = \sigma^2 \left[\begin{array}{cc} 1.854 & 0.618 \\ 0.618 & 1.854 \end{array}\right]$$

Now, setting the vector $\boldsymbol{a} = (-1,1)^\mathsf{T}$ yields that

$$R_{IQ} = X_{(k_2)} - X_{(k_1)} = \boldsymbol{a}^\mathsf{T}(X_{(k_1)}, X_{(k_2)})^\mathsf{T}$$

and hence (by continuous mapping/Slutsky)

$$\sqrt{n}(R_{IQ} - r_{IQ}) \xrightarrow{\mathfrak{L}} N\left(0, \boldsymbol{a}^\mathsf{T}\Sigma\boldsymbol{a}\right)$$

where $r_{IQ} = x_{0.75} - x_{0.25}$. Here

$$r_{IQ} = x_{0.75} - x_{0.25} = 2 \times 0.674\sigma = 1.348\sigma$$

and

$$\boldsymbol{a}^\mathsf{T}\Sigma\boldsymbol{a} = 2.472\sigma^2$$

and hence

$$R_{IQ} \sim AN\left(1.348\sigma, \frac{2.472\sigma^2}{n}\right).$$

[10 MARKS]

2. We have

$$\begin{aligned} d_H(f_1, f_2) &= \int_{-\infty}^{\infty}\left(\sqrt{f_1(x)} - \sqrt{f_2(x)}\right)^2 dx = \int_{-\infty}^{\infty}\left(f_1(x) + f_2(x) - 2\sqrt{f_1(x)f_2(x)}\right) dx \\ &= 2 - 2\int_{-\infty}^{\infty}\sqrt{f_1(x)f_2(x)}\, dx \\ &\leq 2 \end{aligned}$$

as

$$\int_{-\infty}^{\infty}\sqrt{f_1(x)f_2(x)}\, dx \geq 0.$$

[5 MARKS]