RESULT 1: If \( Y_1, Y_2, ..., Y_{n+1} \sim \text{Exponential} (1) \) are independent random variables, and \( S_1, S_2, ..., S_{n+1} \) are defined by

\[
S_k = \sum_{j=1}^{k} Y_j \quad k = 1, 2, ..., n + 1
\]

then the random variables

\[
\left[ \frac{S_1}{S_{n+1}}, \frac{S_2}{S_{n+1}}, ..., \frac{S_n}{S_{n+1}} \right]
\]

given that \( S_{n+1} = s \), say, have the same distribution as the order statistics from a random sample of size \( n \) from the Uniform distribution on \((0, 1)\).

**Proof:** Let the \( Y_j \)'s be defined as above. Then the joint density for the \( Y_j \)'s is given by

\[
\exp \left\{ - \sum_{j=1}^{n+1} y_j \right\} \quad y_1, y_2, ..., y_{n+1} > 0.
\]

Now

\[
\begin{align*}
S_1 &= Y_1 \\
S_2 &= Y_1 + Y_2 \\
S_3 &= Y_1 + Y_2 + Y_3 \\
&\quad \vdots \\
S_n &= \sum_{j=1}^{n} Y_j \\
S_{n+1} &= \sum_{j=1}^{n+1} Y_j
\end{align*}
\]

and so the Jacobian of the transformation from \((Y_1, ..., Y_{n+1}) \rightarrow (S_1, ..., S_{n+1})\) is

\[
\begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \vdots & \vdots & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{vmatrix} = 1
\]

and hence the joint density for \((S_1, ..., S_{n+1})\) is given by

\[
\exp \left\{ -s_{n+1} \right\} \quad 0 < s_1 < s_2 < ... < s_{n+1}.
\]

The marginal distribution for \( S_{n+1} \) is \( \text{Gamma} (n + 1, 1) \) and thus the conditional distribution of \((S_1, ..., S_n)\) given \( S_{n+1} = s \) is

\[
\frac{1}{\Gamma (n + 1)} s^n \exp \left\{ -s \right\} = \frac{n!}{s^n} \quad 0 < s_1 < s_2 < ... < s.
\]

Finally, conditional on \( S_{n+1} = s \), define the joint transformation

\[
V_j = \frac{S_j}{s} \Leftrightarrow S_j = sV_j \quad j = 1, 2, ..., n
\]
which has Jacobian $s^n$. Then, conditional on $S_{n+1} = s$, $(V_1, ..., V_n)$ have joint pdf equal to $n!$ for $0 < v_1 < v_2 < ... < v_n < 1$. Finally, if $U_1, ..., U_n$ are independent random variables each having a Uniform distribution on $(0, 1)$, then $(U_1, ..., U_n)$ have joint pdf equal to 1 on the unit hypercube in $n$ dimensions, and thus the corresponding order statistics $U_{(1)}, ..., U_{(n)}$ also have joint pdf equal to $n! 0 < u_1 < u_2 < ... < u_n < 1$.

RESULT 2: Let the $S_k$ be defined as in Result 1. Then

$$\sqrt{k} \left( \frac{1}{k} S_k - 1 \right) \overset{d}{\to} N(0, 1) \text{ as } k \to \infty$$

Proof: We have that $S_k$ is the sum of $k$ independent and identically distributed Exponential(1) random variables, $Y_1, ..., Y_k$, so that $E[Y_j] = Var[Y_j] = 1$. Thus the Central Limit Theorem applies, and the result follows.

RESULT 3: Let the $S_k$ be defined as in Result 1. Then, if $k_1 n \to p_1$ for some $p_1$ with $0 < p_1 < 1$,

$$\sqrt{n+1} \left( \frac{1}{n+1} S_{k_1} - \frac{k_1}{n+1} \right) \overset{d}{\to} N(0, p_1) \text{ as } n \to \infty$$

Proof: We have

$$\sqrt{n+1} \left( \frac{1}{n+1} S_{k_1} - \frac{k_1}{n+1} \right) = \sqrt{\frac{k_1}{n+1}} \sqrt{k_1} \left( \frac{1}{k_1} S_{k_1} - 1 \right) \overset{d}{\to} \sqrt{p_1} N(0, 1) \equiv N(0, p_1)$$

as $n \to \infty$ (so that by assumption $k_1 \to \infty$ also).

Corollary: Using the same approach, if

$$\frac{k_1}{n} \to p_1 \quad \text{and} \quad \frac{k_2}{n} \to p_2$$

for $0 < p_1 < p_2 < 1$, then

$$\sqrt{n+1} \left( \frac{1}{n+1} (S_{k_2} - S_{k_1}) - \frac{k_2 - k_1}{n+1} \right) = \sqrt{\frac{k_2 - k_1}{n+1}} \sqrt{k_2 - k_1} \left( \frac{1}{k_2 - k_1} \sum_{j=k_1+1}^{k_2} Y_j - 1 \right)$$

and the right-hand side converges in law to $\sqrt{p_2 - p_1} N(0, 1) \equiv N(0, p_2 - p_1)$. Similarly

$$\sqrt{n+1} \left( \frac{1}{n+1} (S_{n+1} - S_{k_2}) - \frac{n+1 - k_2}{n+1} \right) \overset{d}{\to} N(0, 1 - p_2)$$

where the limiting variables in the three cases are independent, as $S_{k_1}, (S_{k_2} - S_{k_1})$, and $(S_{n+1} - S_{k_2})$ are independent.

RESULT 4: Let

$$Z_1 = \frac{1}{n+1} S_{k_1}$$

$$Z_2 = \frac{1}{n+1} (S_{k_2} - S_{k_1})$$

$$Z_3 = \frac{1}{n+1} (S_{n+1} - S_{k_2})$$
and suppose that
\[ \sqrt{n} \left( \frac{k_1}{n} - p_1 \right) \to 0 \text{ and } \sqrt{n} \left( \frac{k_2}{n} - p_2 \right) \to 0 \]
as \( n \to \infty \). Then
\[ \sqrt{n+1} \left( \frac{1}{n+1} S_k - p_1 \right) - \sqrt{n+1} \left( \frac{1}{n+1} S_k - \frac{k_1}{n+1} \right) = \sqrt{n+1} \left( \frac{k_1}{n+1} - p_1 \right) \to 0 \]
as \( n \to \infty \) by assumption, so
\[ \sqrt{n+1} \left( \frac{1}{n+1} S_k - p_1 \right) \text{ and } \sqrt{n+1} \left( \frac{1}{n+1} S_k - \frac{k_1}{n+1} \right) \]
have the same asymptotic distribution, and thus the result follows from Result 3. The proof is similar for the other two terms. Independence (that is, the diagonal nature of \( \Sigma \)) follows from the independence of
\( S_{k_1}, (S_{k_2} - S_{k_1}), \) and \( (S_{n+1} - S_{k_2}) \).

**RESULT 5:** If \( U_{(1)}, \ldots, U_{(n)} \) are the order statistics from a random sample of size \( n \) from a \( Uniform(0,1) \) distribution, and if \( n \to \infty, k_1 \to \infty \) and \( k_2 \to \infty \) in such a way that
\[ \sqrt{n} \left( \frac{k_1}{n} - p_1 \right) \to 0 \text{ and } \sqrt{n} \left( \frac{k_2}{n} - p_2 \right) \to 0 \]
for \( 0 < p_1 < p_2 < 1 \), then
\[ \sqrt{n} \left( \begin{pmatrix} U_{(k_1)} \\ U_{(k_2)} \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right) \overset{\mathcal{L}}{\to} N \left( 0, \begin{bmatrix} p_1 (1-p_1) & p_1 (1-p_2) \\ p_1 (1-p_2) & p_2 (1-p_2) \end{bmatrix} \right). \]

**Proof:** Define
\[ g(x_1, x_2, x_3) = \frac{1}{x_1 + x_2 + x_3} \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix} \]
which yields first derivative
\[ \dot{g}(x_1, x_2, x_3) = \frac{1}{(x_1 + x_2 + x_3)^2} \begin{bmatrix} x_2 + x_3 & -x_1 & -x_1 \\ x_3 & x_3 & -(x_1 + x_2) \end{bmatrix}. \]
Now
\[ g \left( \frac{S_{k_1}}{n+1}, \frac{S_{k_2} - S_{k_1}}{n+1}, \frac{S_{n+1} - S_{k_2}}{n+1} \right) = \frac{1}{S_{n+1}} \begin{bmatrix} S_{k_1} \\ S_{k_2} \end{bmatrix} \]
which has the same distribution as \( (U_{(k_1)}, U_{(k_2)})^T \), by Result 1. By Cramer’s Theorem
\[ \sqrt{n} \left( \begin{pmatrix} U_{(k_1)} \\ U_{(k_2)} \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right) \overset{\mathcal{L}}{\to} N \left( 0, \dot{g}(\mu) \Sigma \dot{g}(\mu)^T \right) \]
where \( \Sigma \) is as defined in the Result 4, where here \( \mu = (p_1, p_2 - p_1, 1 - p_2)^T \). It can be easily verified that
\[ \dot{g}(\mu) \Sigma \dot{g}(\mu)^T = \begin{bmatrix} p_1 (1-p_1) & p_1 (1-p_2) \\ p_1 (1-p_2) & p_2 (1-p_2) \end{bmatrix} \]
and thus the result follows.
RESULT 6: If $X_1, \ldots, X_n$ are the order statistics from a random sample of size $n$ from a distribution with continuous distribution function $F_X$ and density $f_X$ which is continuous and non-zero in a neighbourhood of quantiles $x_{p_1}$ and $x_{p_2}$ corresponding to probabilities $p_1 < p_2$, then if $k_1 = \lceil np_1 \rceil$ and $k_2 = \lceil np_2 \rceil$

$$\sqrt{n} \left( \begin{pmatrix} X_{(k_1)} \\ X_{(k_2)} \end{pmatrix} - \begin{pmatrix} x_{p_1} \\ x_{p_2} \end{pmatrix} \right) \xrightarrow{d} N \left( \begin{pmatrix} p_1 (1 - p_1) \\ p_2 (1 - p_2) \end{pmatrix}, \begin{pmatrix} \frac{f_X(x_{p_1})^2}{f_X(x_{p_1}) f_X(x_{p_2})} & \frac{p_1 (1 - p_2)}{f_X(x_{p_1}) f_X(x_{p_2})} \\ \frac{p_1 (1 - p_2)}{f_X(x_{p_1}) f_X(x_{p_2})} & \frac{p_2 (1 - p_2)}{f_X(x_{p_2})^2} \end{pmatrix} \right)$$

Proof: We use the Delta Method (Cramer’s Theorem) on the result from Result 5, with the transformation

$$g(y_1, y_2) = \begin{bmatrix} F_X^{-1}(y_1) \\ F_X^{-1}(y_2) \end{bmatrix}$$

so that

$$\dot{g}(y_1, y_2) = \begin{bmatrix} \frac{1}{f_X(F_X^{-1}(y_1))} & 0 \\ 0 & \frac{1}{f_X(F_X^{-1}(y_2))} \end{bmatrix}$$

with $y_1 = p_1$ and $y_2 = p_2$. 