MEASURABLE FUNCTIONS

The real-valued function $f$ defined with domain $E \subset \Omega$, for measurable space $(\Omega, \mathcal{F})$, is **Borel measurable** with respect to $\mathcal{F}$ if the inverse image of set $B$, defined as

$$f^{-1}(B) \equiv \{\omega \in E : f(\omega) \in B\}$$

is an element of $\sigma$-algebra $\mathcal{F}$, for all Borel sets $B$ of $\mathbb{R}$ (strictly, of the extended real number system $\mathbb{R}^*$, including $\pm \infty$ as elements). The following conditions are each necessary and sufficient for $f$ to be measurable

(a) $f^{-1}(A) \in \mathcal{F}$ for all open sets $A \subset \mathbb{R}^*$,
(b) $f^{-1}([\infty, x)) \in \mathcal{F}$ for all $x \in \mathbb{R}^*$,
(c) $f^{-1}([\infty, x]) \in \mathcal{F}$ for all $x \in \mathbb{R}^*$,
(d) $f^{-1}([x, \infty)) \in \mathcal{F}$ for all $x \in \mathbb{R}^*$,
(e) $f^{-1}((x, \infty]) \in \mathcal{F}$ for all $x \in \mathbb{R}^*$.

NOTES:

(i) The **Borel $\sigma$-algebra** in $\mathbb{R}$, $\mathcal{B}$, is the smallest (or **minimal**) $\sigma$-algebra containing all open sets (that is, essentially, sets of the form $(a, b)$ or $[a, b)$ for $a < b \in \mathbb{R}$) which are known as the **Borel sets** in $\mathbb{R}$.

(ii) It is possible to extend this definition to a general topological space $\Omega$ equipped with a topology, that is, a collection, $\mathcal{T}$, of sets in $\Omega$ that (I) $\mathcal{T}$ contains $\emptyset$ and $\Omega$, (II) $\mathcal{T}$ is closed under finite intersection, and (III) if $\mathcal{A}$ is a sub-collection of $\mathcal{T}$, $\mathcal{A} \subset \mathcal{T}$, and $A_1, A_2, A_3, \ldots \in \mathcal{A}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{T}.$$ 

In this context, it is possible to define a general Borel $\sigma$-algebra on $\Omega$; the open sets are the elements $T_1, T_2, T_3, \ldots$ of the topology $\mathcal{T}$, and the Borel sets are the elements of the smallest $\sigma$-algebra generated by $\mathcal{T}$, $\sigma(\mathcal{T})$. However, we will not be studying general topological spaces; we shall restrict attention to $\mathbb{R}$, and thus refer to the Borel sets and the Borel $\sigma$-algebra, meaning the Borel sets/$\sigma$-algebra defined on $\mathbb{R}$.

(iii) Strictly, a function $f$ is a **Borel function** if, for $B \in \mathcal{B}$, $f^{-1}(B) \in \sigma(\mathcal{T})$; however, we will generally consider measure spaces $(\Omega, \mathcal{F})$ and say that $f$ is a **Borel function** if it is Borel measurable, as defined in the first paragraph above.
Example Consider Lebesgue measure, $m$, defined for real numbers $a < b$ (on the Borel $\sigma$-algebra on $\mathbb{R}$, $\mathcal{B}$) by

$$m ([a, b]) = m ((a, b]) = m ([a, b)) = m ((a, b)) = b - a.$$ 

Suppose $f$ is an increasing function on $\mathbb{R}$. Then the set $A \equiv f^{-1} (]-\infty, x])$ is an interval in $\mathbb{R}$, and thus $f$ is measurable with respect to Lebesgue measure, as the measure of $A$, $m(A)$, is well-defined. Now consider the function $g$ defined by $g(x) = x$ for $x \in \mathbb{R}$. This function is measurable with respect to Lebesgue measure (on $\mathcal{B}$), as it is increasing. However, consider the $\sigma$-algebra, $\mathcal{Z}$, generated by the sets $\{0, (-\infty, 0], (0, \infty), \mathbb{R}\}$. Then

$$g^{-1} (]-\infty, 1]) \notin \mathcal{Z}$$

so $g$ is not measurable on $\mathcal{Z}$.

### RESULTS FOR MEASURABLE FUNCTIONS

**Theorem 1.1 MEASURABILITY UNDER COMPOSITION**

Let $g_1$ and $g_2$ be measurable functions on $E \subset \Omega$ with ranges in $\mathbb{R}^1$. Let $f$ be a Borel function from $\mathbb{R}^1 \times \mathbb{R}^1$ into $\mathbb{R}$. Then the composite function $h$, defined on $E$ by

$$h(\omega) = f(g_1(\omega_1), g_2(\omega_2))$$

is measurable.

**Proof.** The function $g = (g_1, g_2)$ has domain $E$ and range $\mathbb{R}^1 \times \mathbb{R}^1$, and is measurable as $g_1$ and $g_2$ are measurable, and denote $h = f \circ g$ (the operator $\circ$ indicates composition, i.e.

$$h(\omega_1, \omega_2) = (f \circ g)(\omega_1, \omega_2) \quad \text{if} \quad h(\omega_1, \omega_2) = f(g(\omega_1, \omega_2)) = f(g_1(\omega_1), g_2(\omega_2)).$$

If $B \in \mathcal{B}$, then $f^{-1}(B)$ is a Borel set as $f$ is a Borel function. Thus the inverse image under $h$,

$$h^{-1}(B) = g^{-1}(f^{-1}(B))$$

is measurable as $g_1$ and $g_2$, and hence $g$, are measurable.

**Corollary** If $g$ is a measurable function from $E$ into $\mathbb{R}^1$, and $f$ is a continuous function from $\mathbb{R}^1$ into $\mathbb{R}^1$, then $h = f \circ g$ is measurable.

**Theorem 1.2 MEASURABILITY UNDER ELEMENTARY OPERATIONS**

Let $g_1$ and $g_2$ be measurable functions defined on $E \subset \Omega$ into $\mathbb{R}^1$, and let $c$ be any real number. Then all of the following composite and other related functions are measurable

$$g_1 + g_2, g_1 + c, g_1 g_2, c g_1, g_1 / g_2, |g_1|, g_1 \vee g_2, g_1 \wedge g_2, g_1^+, g_1^-.$$ 

**Proof.** In each case, we examine the domain of the composite function to ensure measurability in the Borel $\sigma$-algebra. Consider $g_1 + g_2$; this is not defined on the set

$$\{\omega : g_1(\omega) = -g_2(\omega) = \pm \infty\}$$

(as $\infty \pm \infty$ is not defined), but this set is measurable, and so is the domain of $g_1 + g_2$. Let $f(x_1, x_2) = x_1 + x_2$ be a continuous function defined on $\mathbb{R}^1 \times \mathbb{R}^1$. Then, by Theorem 1.1 and its corollary, $g_1 + g_2$ is measurable. Taking $g_2 = c$ proves that $g_1 + c$ is measurable.

The function $g_1 g_2$ is defined everywhere on $E$; it’s measurability follows from Theorem 1.1,
setting \( f(x_1, x_2) = x_1 x_2 \). Setting \( g_2 = c \) proves that \( c g_1 \) is measurable.

The function \( g_1 / g_2 \) is defined everywhere except on the union of sets

\[
\{ \omega : g_1(\omega) = g_2(\omega) = 0 \} \cup \{ \omega : \pm g_1(\omega) = \pm g_2(\omega) = \infty \}
\]

Similarly, if \( c = 0 \), \( |g_1| \) is defined except on

\[
\{ \omega : g_1(\omega) = \pm \infty \}
\]

if \( c < 0 \), it is defined except on

\[
\{ \omega : g_1(\omega) = 0 \}
\]

If \( c > 0 \), it is defined everywhere. All of these sets are measurable. Thus, we consider in turn functions

\[
f(x_1, x_2) = x_1 / x_2 \quad f(x) = x^c
\]

and use Theorem 1.1.

The functions \( g_1 \lor g_2, g_1 \land g_2 \) are defined everywhere; so we consider functions

\[
f(x_1, x_2) = \max \{x_1, x_2\} \quad f(x_1, x_2) = \min \{x_1, x_2\}
\]

and again use Theorem 1.1. Finally, setting \( g_2 = 0 \) yields the measurability of \( g_1^+ \) and \( g_1^- \).

**Theorem 1.3** If \( g_1 \) and \( g_2 \) are measurable functions on a common domain, then each of the sets

\[
\{ \omega : g_1(\omega) < g_2(\omega) \} \quad \{ \omega : g_1(\omega) = g_2(\omega) \} \quad \{ \omega : g_1(\omega) > g_2(\omega) \}
\]

is measurable.

**Proof.** Since \( g_1 \) and \( g_2 \) are measurable, then \( f = g_1 - g_2 \) is measurable, and thus the two sets

\[
\{ \omega : f(\omega) > 0 \} \quad \{ \omega : f(\omega) = 0 \}
\]

are measurable. Since

\[
\{ \omega : g_1(\omega) < g_2(\omega) \} \equiv \{ \omega : f(\omega) > 0 \}
\]

and

\[
\{ \omega : g_1(\omega) = g_2(\omega) \} \equiv \{ \omega : f(\omega) = 0 \} \cup \{ \omega : g_1(\omega) = g_2(\omega) = \pm \infty \}
\]

then \( \{ \omega : g_1(\omega) < g_2(\omega) \} \) and \( \{ \omega : g_1(\omega) = g_2(\omega) \} \) are measurable, and so is

\[
\{ \omega : g_1(\omega) \leq g_2(\omega) \} \equiv \{ \omega : g_1(\omega) < g_2(\omega) \} \cup \{ \omega : g_1(\omega) = g_2(\omega) \}.
\]
Theorem 1.4 MEASURABILITY UNDER LIMIT OPERATIONS
If \( \{g_n\} \) is a sequence of measurable functions, the functions \( \sup_n g_n \) and \( \inf_n g_n \) are measurable.

**Proof.** Let \( g = \sup_n g_n \). Then for real \( x \), consider
\[
 g_n^{-1}([-\infty, x]) \equiv \{ \omega : g_n(\omega) \leq x \}
\]
and
\[
 g^{-1}([-\infty, x]) \equiv \{ \omega : g(\omega) \leq x \}.
\]
If \( g = \sup_n g_n \), then \( g_n \leq g \) for all \( n \), and
\[
 g(\omega) \leq x \implies g_n(\omega) \leq x \quad \text{so that} \quad \omega \in g^{-1}([-\infty, x]) \implies \omega \in g_n^{-1}([-\infty, x])
\]
so that
\[
 g^{-1}([-\infty, x]) \subseteq g_n^{-1}([-\infty, x])
\]
for all \( n \). Thus, in fact
\[
 g^{-1}([-\infty, x]) = \bigcap_n g_n^{-1}([-\infty, x])
\]
and hence \( g \) is measurable, as the intersection of measurable sets is measurable. The result for \( \inf_n \) follows by noting that
\[
 \inf_n g_n = -\sup_n (-g_n).
\]

Theorem 1.5 MEASURABILITY UNDER LIMINF/LIMSUP
If \( \{g_n\} \) is a sequence of measurable functions, the functions \( \limsup_n g_n \) and \( \liminf_n g_n \) are measurable.

**Proof.** This follows from Theorem 1.4, as
\[
 \limsup_n g_n = \inf_k \left\{ \sup_{n \geq k} g_n \right\} \quad \text{and} \quad \liminf_n g_n = \sup_k \left\{ \inf_{n \geq k} g_n \right\}
\]

SIMPLE FUNCTIONS AND THEIR CONVERGENCE PROPERTIES.

**Definition:** Simple Functions
A *simple function*, \( \psi \), is a set function defined on elements \( \omega \) of sample space \( \Omega \) by
\[
 \psi(\omega) = \sum_{i=1}^{k} a_i I_{A_i}(\omega)
\]
for real constants \( a_1, ..., a_k \) and measurable sets \( A_1, ..., A_k \), for some \( k = 1, 2, 3, ..., \), where \( I_A(\omega) \) is the *indicator function*, where
\[
 I_A(\omega) = \begin{cases} 
 1 & \omega \in A \\
 0 & \omega \notin A
\end{cases}.
\]
Note that any such simple function, can be re-expressed as a simple function defined for a *partition* of \( \Omega \), \( E_1, ..., E_l \),
\[
 \psi(\omega) = \sum_{i=1}^{l} e_i I_{E_i}(\omega)
\]
by suitable choice of the constants \( e_1, ..., e_k \).
Theorem 1.6 A non-negative function on $\Omega$ is measurable if and only if it is the limit of an increasing sequence of non-negative simple functions.

Proof. Suppose that $g$ is a nonnegative measurable function. For each positive integer $n$, define the simple function $\psi_n$ on $\Omega$ by

$$
\psi_n(\omega) = \frac{m}{2^n} \quad \text{if} \quad \frac{m}{2^n} \leq g(\omega) < \frac{m+1}{2^n}
$$

for $m = 0, 1, 2, \ldots, 2^n - 1$, and

$$
\psi_n(\omega) = n \quad \text{if} \quad n \leq g(\omega).
$$

Then $\{\psi_n\}$ is an increasing sequence of non-negative simple functions. Since

$$
|\psi_n(\omega) - g(\omega)| < \frac{1}{2^n} \quad \text{if} \quad n > g(\omega)
$$

and $\psi_n(\omega) = n$ if $g(\omega) = \infty$, then, for all $\omega$,

$$
\psi_n(\omega) \to g(\omega)
$$

and we have found the sequence required for the result.

Now suppose that $g$ is a limit of an increasing sequence of non-negative simple functions. Then it is measurable by Theorem 1.5.

Theorem 1.7 A function $g$ defined on $\Omega$ is measurable if and only if it is the limit of a sequence of simple functions.

Proof. Suppose that $g$ is measurable. Then $g^+$ and $g^-$ are measurable and non-negative, and thus can be represented as limits of simple functions $\{\psi_n^+\}$ and $\{\psi_n^-\}$, by the Theorem 1.6. Consider the sequence of simple functions defined by $\{\psi_n^+ - \psi_n^-\}$; this sequence converges to $g^+ - g^- = g$, and we have the sequence of simple functions required for the result.

Now suppose that $g$ is a limit of a sequence of simple functions. Then it is measurable by Theorem 1.5.