M2S1 : EXERCISE SHEET 6 : SOLUTIONS

1. We have the marginal of X given in the usual way from the joint density by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^{\infty} f_{X|Y}(x|y) f_Y(y) dy \qquad x > 0$$

as we have the density being positive only when arguments x and y are positive. Hence

$$\begin{aligned} f_X(x) &= \int_0^{\infty} f_{X|Y}(x|y) f_Y(y)\ dy = \int_0^{\infty} y e^{-yx} \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}\ dy = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{\infty} y^{(\alpha+1)-1} e^{-(\beta+x)y} dy \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(\beta+x)^{\alpha+1}} = \frac{\alpha\beta^\alpha}{(\beta+x)^{\alpha+1}} \qquad x > 0 \qquad \text{as } \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \end{aligned}$$

(integrand is proportional.to a $Gamma(\alpha+1, \beta+x)$ pdf). Hence $X \sim Pareto(\alpha, \beta)$

2. To compute the joint density f_{Y_1,Y_2}, use the multivariate transformation theorem; we have

$$\left. \begin{aligned} Y_1 &= \mu_1 + \sigma_1\sqrt{1-\rho^2} X_1 + \sigma_1 \rho X_2 \\ Y_2 &= \mu_2 + \sigma_2 X_2 \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} X_1 &= (Y_1 - \mu_1)/(\sigma_1\sqrt{1-\rho^2}) - \rho(Y_2 - \mu_2)/\left(\sigma_2\sqrt{1-\rho^2}\right) \\ X_2 &= (Y_2 - \mu_2)/\sigma_2 \end{aligned} \right.$$

and hence the Jacobian $J(y_1, y_2)$ is the modulus of the determinant of the matrix of partial derivatives;

$$J(y_1, y_2) = \left| \left| \begin{bmatrix} \dfrac{1}{\sigma_1\sqrt{1-\rho^2}} & \dfrac{-\rho}{\sigma_2\sqrt{1-\rho^2}} \\ 0 & \dfrac{1}{\sigma_2} \end{bmatrix} \right| \right| = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

as σ_1, σ_2, and $\sqrt{1-\rho^2}$ are all positive quantities. Hence the joint pdf f_{Y_1,Y_2} is given in terms of the joint pdf $f_{X_1,X_2}$by

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2) J(y_1, y_2) = \frac{1}{2\pi} \exp\left\{ -\frac{1}{2}\left(x_1^2 + x_2^2\right) \right\} J(y_1, y_2)$$

where

$$x_1 = (y_1 - \mu_1)/(\sigma_1\sqrt{1-\rho^2}) - \rho(y_2 - \mu_2)/(\sigma_2\sqrt{1-\rho^2}) \qquad x_2 = (y_2 - \mu_2)/\sigma_2$$

for fixed (y_1, y_2) define the inverse transformations. Now,

$$\begin{aligned} x_1^2 + x_2^2 &= \left((y_1 - \mu_1)/(\sigma_1\sqrt{1-\rho^2}) - \rho(y_2 - \mu_2)/(\sigma_2\sqrt{1-\rho^2})\right)^2 + ((y_2 - \mu_2)/\sigma_2)^2 \\ &= \frac{(y_1 - \mu_1)^2}{\sigma_1^2(1-\rho^2)} + \frac{\rho^2 (y_2 - \mu_2)^2}{\sigma_2^2(1-\rho^2)} - \frac{2\rho(y_1 - \mu_1)(y_2 - \mu_2)}{\sigma_1\sigma_2(1-\rho^2)} + \frac{(y_2 - \mu_2)^2}{\sigma_2^2} \\ &= \frac{1}{(1-\rho^2)}\left[\frac{(y_1 - \mu_1)^2}{\sigma_1^2} + \frac{(y_2 - \mu_2)^2}{\sigma_2^2} - \frac{2\rho(y_1 - \mu_1)(y_2 - \mu_2)}{\sigma_1\sigma_2} \right] \end{aligned}$$

and hence using the transformation result we have

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}((y_1 - \mu_1)/(\sigma_1\sqrt{1-\rho^2}) - \rho(Y_2 - \mu_2)/(\sigma_2\sqrt{1-\rho^2}), (y_2 - \mu_2)/\sigma_2) J(y_1, y_2) \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(y_1 - \mu_1)^2}{\sigma_1^2} + \frac{(y_2 - \mu_2)^2}{\sigma_2^2} - \frac{2\rho(y_1 - \mu_1)(y_2 - \mu_2)}{\sigma_1\sigma_2} \right] \right\} \end{aligned}$$

This is the Bivariate Normal pdf; we say that (Y_1, Y_2) have a bivariate normal distribution. Note that this function is symmetric in form; we can exchange the triples (y_1, μ_1, σ_1) and (y_2, μ_2, σ_2) without changing the functional form. Note finally that, in vector form we have the pdf in the form

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{y} - \boldsymbol{\mu})\right\} \qquad \mathbf{y} \in \mathbb{R}^2$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \qquad \Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

Now, could compute the marginal pdf of Y_1 and Y_2 by integrating out from the joint pdf, for example

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2 \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{1}{\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2}\right]\right\} dy_2 \end{aligned}$$

Setting $t_1 = (y_1 - \mu_1)/\sigma_1$ (*a constant*), and substituting $t_2 = (y_2 - \mu_2)/\sigma_2$ in this integral we obtain

$$f_{Y_1}(y_1) = \frac{1}{2\pi} \frac{1}{\sqrt{\sigma_1^2(1-\rho^2)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[t_1^2 + t_2^2 - 2\rho t_1 t_2\right]\right\} dt_2$$

and can complete the square in the exponent as $t_1^2 + t_2^2 - 2\rho t_1 t_2 = (t_2 - \rho t_1)^2 + t_1^2(1-\rho^2)$, so that

$$\begin{aligned} f_{Y_1}(y_1) &= \frac{1}{2\pi} \frac{1}{\sqrt{\sigma_1^2(1-\rho^2)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[(t_2-\rho t_1)^2 + t_1^2(1-\rho^2)\right]\right\} dt_2 \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{\sigma_1^2(1-\rho^2)}} \exp\left\{-\frac{1}{2}t_1^2\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{(t_2-\rho t_1)^2}{2(1-\rho^2)}\right\} dt_2 \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{\sigma_1^2(1-\rho^2)}} \exp\left\{-\frac{1}{2}t_1^2\right\} \sqrt{2\pi(1-\rho^2)} \end{aligned}$$

as the integrand is proportional to a Normal pdf with expectation ρt_1 and variance $(1-\rho^2)$. Therefore, cancelling terms and substituting back in for $y_1$we have

$$f_{Y_1}(y_1) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left\{-\frac{1}{2}\frac{(y_1-\mu_1)^2}{\sigma_1^2}\right\}$$

so that $Y_1 \sim Normal(\mu_1, \sigma_1^2)$. By symmetry, we have that $Y_2 \sim Normal(\mu_2, \sigma_2^2)$.

Note also that we have that $Y_2 = \mu_2 + \sigma_2 X_2$ implies $Y_2 \sim Normal(\mu_2, \sigma_2^2)$ from elementary properties (location/scale transformations) of Normal random variables. For the conditional distributions, can use elementary properties of Normal random variables again, that is, *given* $Y_2 = y_2$ so that $X_2 = (y_2 - \mu_2)/\sigma_2$

$$Y_1 = \mu_1 + \sigma_1\sqrt{1-\rho^2}X_1 + \sigma_1\rho(y_2-\mu_2)/\sigma_2 \sim Normal(\mu_1 + \sigma_1\rho(y_2-\mu_2)/\sigma_2, \sigma_1^2(1-\rho^2))$$

that is, via a location/scale transformation $Y_1 = a + bX_1$with $a = \mu_1 + \sigma_1\rho(y_2-\mu_2)/\sigma_2$ and $b = \sigma_1\sqrt{1-\rho^2}$, and similarly for the conditional for Y_2 given $Y_1 = y_1$. Note that the conditional densities can also be computed from the definition

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_2}(y_2)}$$

To compute the correlation, first compute the covariance using the Iterated Expectation result; we calculate

$$\mathrm{Cov}_{f_{Y_1},f_{Y_2}}[Y_1,Y_2] = \mathrm{E}_{f_{Y_1},f_{Y_2}}[Y_1Y_2] - \mathrm{E}_{f_{Y_1}}[Y_1]\,\mathrm{E}_{f_{Y_2}}[Y_2]$$

where, by the Law of Iterated Expectation

$$\mathrm{E}_{f_{Y_1},f_{Y_2}}[Y_1Y_2] = \mathrm{E}_{f_{Y_2}}\left[\mathrm{E}_{f_{Y_1|Y_2}}[Y_1Y_2|Y_2=y_2]\right]$$

and as $Y_1|Y_2 = y_2 \sim Normal(\mu_1 + \sigma_1\rho(y_2-\mu_2)/\sigma_2, \sigma_1^2(1-\rho^2))$

$$\mathrm{E}_{f_{Y_1|Y_2}}[Y_1Y_2|Y_2=y_2] = (\mu_1 + \sigma_1\rho(y_2-\mu_2)/\sigma_2)\,y_2$$

and hence

$$\begin{aligned}
\mathrm{E}_{f_{Y_2}}\left[\mathrm{E}_{f_{Y_1|Y_2}}[Y_1Y_2|Y_2=y_2]\right] &= \mathrm{E}_{f_{Y_2}}[(\mu_1 + \sigma_1\rho(Y_2-\mu_2)/\sigma_2)\,Y_2] \\
&= (\mu_1 - \sigma_1\rho\mu_2/\sigma_2)\mathrm{E}_{f_{Y_2}}[Y_2] + \sigma_1\rho\mathrm{E}_{f_{Y_2}}\left[Y_2^2\right]/\sigma_2 \\
&= (\mu_1 - \sigma_1\rho\mu_2/\sigma_2)\mu_2 + \sigma_1\rho(\mu_2^2+\sigma_2^2)/\sigma_2 \\
&= \mu_1\mu_2 - \sigma_1\rho\mu_2^2/\sigma_2 + \sigma_1\rho\mu_2^2/\sigma_2 + \sigma_1\sigma_2\rho
\end{aligned}$$

and hence

$$\mathrm{E}_{f_{Y_1},f_{Y_2}}[Y_1Y_2] = \mu_1\mu_2 + \sigma_1\sigma_2\rho$$

$$\mathrm{Cov}_{f_{Y_1},f_{Y_2}}[Y_1,Y_2] = \mathrm{E}_{f_{Y_1},f_{Y_2}}[Y_1Y_2] - \mathrm{E}_{f_{Y_1}}[Y_1]\,\mathrm{E}_{f_{Y_2}}[Y_2] = \mu_1\mu_2 + \sigma_1\sigma_2\rho - \mu_1\mu_2 = \sigma_1\sigma_2\rho$$

so that, finally,

$$\mathrm{Corr}_{f_{Y_1},f_{Y_2}}[Y_1,Y_2] = \frac{\mathrm{Cov}_{f_{Y_1},f_{Y_2}}[Y_1,Y_2]}{\sqrt{\mathrm{Var}_{f_{Y_1}}[Y_1]\,\mathrm{Var}_{f_{Y_2}}[Y_2]}} = \frac{\sigma_1\sigma_2\rho}{\sqrt{\sigma_1^2\sigma_2^2}} = \rho$$

3. We have, for the inverse transformations

$$\left.\begin{aligned} Z_1 &= \sqrt{-2\log U_1}\cos(2\pi U_2) \\ Z_2 &= \sqrt{-2\log U_1}\sin(2\pi U_2)\end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} U_1 &= \exp\left\{-\frac{1}{2}\left(Z_1^2+Z_2^2\right)\right\} \\ U_2 &= \frac{1}{2\pi}\arctan\frac{Z_2}{Z_1}\end{aligned}\right. .$$

The range of the new variables is $\mathbb{R}\times\mathbb{R}$. The Jacobian of the transformation $(U_1,U_2)\to(Z_1,Z_2)$ is

$$\begin{aligned}
\begin{vmatrix} \dfrac{\partial u_1}{\partial z_1} & \dfrac{\partial u_1}{\partial z_2} \\ \dfrac{\partial u_2}{\partial z_1} & \dfrac{\partial u_2}{\partial z_2}\end{vmatrix} &= \begin{vmatrix} z_1\exp\left\{-\frac{1}{2}\left(z_1^2+z_2^2\right)\right\} & z_2\exp\left\{-\frac{1}{2}\left(z_1^2+z_2^2\right)\right\} \\ -\dfrac{1}{2\pi}\dfrac{z_2}{z_1^2+z_2^2} & \dfrac{1}{2\pi}\dfrac{z_1}{z_1^2+z_2^2}\end{vmatrix} \\
&= \left|\frac{1}{2\pi}\frac{z_1^2}{z_1^2+z_2^2}\exp\left\{-\frac{1}{2}\left(z_1^2+z_2^2\right)\right\} + \frac{1}{2\pi}\frac{z_2^2}{z_1^2+z_2^2}\exp\left\{-\frac{1}{2}\left(z_1^2+z_2^2\right)\right\}\right| \\
&= \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(z_1^2+z_2^2\right)\right\}
\end{aligned}$$

Hence the joint pdf is

$$\begin{aligned} f_{Z_1,Z_2}(z_1,z_2) &= f_{U_1,U_2}\left(\exp\left\{-\frac{1}{2}\left(z_1^2+z_2^2\right)\right\}, \frac{1}{2\pi}\arctan\frac{z_2}{z_1}\right) J(z_1,z_2) \\ &= 1 \times \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(z_1^2+z_2^2\right)\right\} = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left(z_1^2+z_2^2\right)\right\}. \end{aligned}$$

for $(z_1, z_2) \in \mathbb{R}^2$. Note that

$$f_{Z_1,Z_2}(z_1,z_2) = f_{Z_1}(z_1)\, f_{Z_2}(z_2)$$

where

$$f_{Z_1}(z_1) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}z_1^2\right\} \qquad f_{Z_2}(z_2) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}z_2^2\right\}$$

so, in fact Z_1 and Z_2 are independent standard Normal random variables.

4. From first principles

$$F_X(x) = P[X \le x] = P[-\beta \log U \le x] = P\left[U \ge \exp\left\{-\frac{x}{\beta}\right\}\right] = 1 - F_U\left(\exp\left\{-\frac{x}{\beta}\right\}\right).$$

But $U \sim Uniform(0,1)$, so $F_U(u) = u$ for $0 < u < 1$, so

$$F_X(x) = 1 - \exp\left\{-\frac{x}{\beta}\right\}$$

and so $X \sim \mathit{Exponential}(1/\beta)$.

(i) sum k *Exponential*(λ) variables $X_1, ..., X_k$, generated independently using the transformed uniform random variables $U_1, ..., U_k$ where

$$X_i = -\frac{1}{\lambda}\log U_i$$

(ii) events in a *Poisson process* with rate μ.can be obtained by taking cumulative sums of the independent exponential random variables from part (i):

$$T_i = \sum_{j=1}^{i} X_j \qquad X_j = -\frac{1}{\mu}\log U_j \ \text{ with } U_j \sim Uniform(0,1)$$

(iii) ν is an integer, by definition of the Chi-squared distribution, and we have that if $Z \sim N(0,1)$, then $X = Z^2 \sim \chi_1^2$. But also, using the addition result for independent Gamma random variables we have that

$$Z_1, ..., Z_\nu \sim N(0,1) \qquad \Longrightarrow Y = \sum_{i=1}^{\nu} Z_i^2 \sim \chi_\nu^2$$

We can simulate Normal random variables using the method from question 3.

(iv) By the result from lectures, simulate

$$Z \sim N(0,1) \text{ and } V \sim \chi_n^2$$

independently using the previously described methods, then take

$$T = \frac{Z}{\sqrt{V/n}}$$

which is a *Student*(n) random variable.