# M2S1 : EXERCISE SHEET 5 : SOLUTIONS

1. To compute the covariance need first the marginal expectations of $X$ and $Y$. The key part of the solution is to realize that the support of the joint density is

$$0 < x < 1, 0 < y < 1, 0 < x + y < 1$$

that is, the "lower left corner" triangle of the unit square, bounded by the three lines $x = 0, y = 0, x + y = 1$. Now, for $0 < x < 1$,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^{1-x} cxy(1-x-y)\ dy = cx\int_0^{1-x} y(1-x-y)\ dy \\ &= cx(1-x)^3 \int_0^1 t(1-t)\ dt \qquad (t = y/(1-x)) \\ &= \frac{c}{6}x(1-x)^3 \qquad 0 < x < 1 \end{aligned}$$

and

$$\int_{-\infty}^{\infty} f_X(x)\ dx = \int_0^1 \frac{c}{6}x(1-x)^3\ dx = 1 \Longrightarrow c = 120$$

and hence

$$f_X(x) = 20x(1-x)^3 \qquad 0 < x < 1 \qquad \therefore \mathrm{E}_{f_X}[\ X\ ] = \int_0^1 20x^2(1-x)^3\ dx = \frac{1}{3}$$

and, by symmetry, $f_Y(y) = 20y(1-y)^3$ $(0 < y < 1)$, $\mathrm{E}_{f_Y}[\ Y\ ] = \dfrac{1}{3}$ by symmetry of form of the joint pdf. Also

$$\begin{aligned} \mathrm{E}_{f_{X,Y}}[\ XY\ ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)\ dxdy = \int_0^1 \left\{\int_0^{1-y} 120x^2y^2(1-x-y)\ dx\right\} dy \\ &= \int_0^1 120y^2 \left\{\int_0^{1-y} x^2(1-x-y)\ dx\right\} dy \\ &= \int_0^1 120y^2 \left[\frac{x^3}{3}(1-y) - \frac{x^4}{4}\right]_0^{1-y} dy \\ &= \int_0^1 10y^2(1-y)^4\ dy \\ &= 10\left[\frac{y^3}{3} - y^4 + \frac{6y^5}{5} - \frac{4y^6}{6} + \frac{y^7}{7}\right]_0^1 = 10\left(\frac{1}{3} - 1 + \frac{6}{5} - \frac{2}{3} + \frac{1}{7}\right) = \frac{2}{21} \end{aligned}$$

and hence

$$\mathrm{Cov}_{f_{X,Y}}[\ X,Y\ ] = \mathrm{E}_{f_{X,Y}}[\ XY\ ] - \mathrm{E}_{f_X}[\ X\ ]\ .\ \mathrm{E}_{f_Y}[\ Y\ ] = \frac{2}{21} - \frac{1}{3}\cdot\frac{1}{3} = -\frac{1}{63}$$

2. (a) Put $U = X/Y$ and $Z = X$; the inverse transformations are therefore $X = Z$ and $Y = Z/U$, and note that the new variables are constrained by $0 < Z < \min\{U, 1\}$, as $Y < 1$. In terms of the multivariate transformation theorem, we have transformation functions defined by

$$g_1(t_1, t_2) = t_1/t_2 \qquad g_1^{-1}(t_1, t_2) = t_2$$

$$g_2(t_1, t_2) = t_1 \qquad g_2^{-1}(t_1, t_2) = t_2/t_1$$

and the Jacobian of the transformation is given by

$$J(u, z) = \begin{vmatrix} 0 & 1 \\ -z/u^2 & 1/u \end{vmatrix} = \frac{z}{u^2}$$

and hence

$$f_{U,Z}(u, z) = f_{X,Y}(z, z/u)\ z/u^2 = z/u^2 \qquad (u, z) \in \mathbb{U}^{(2)} = \{(u, z) : 0 < z < \min\{u, 1\}, u > 0\}$$

and zero otherwise, and so

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,Z}(u, z)\ dz = \int_0^{\min\{u,1\}} z/u^2\ dz = \frac{(\min\{u, 1\})^2}{2u^2} \qquad u > 0.$$

Put $V = -\log(XY)$ and $Z = -\log X$; the inverse transformations are therefore $X = e^{-Z}$ and $Y = e^{-(v-z)}$, and note that $0 < Z < V$. In terms of the theorem, we have transformation functions defined by

$$g_1(t_1, t_2) = -\log(t_1 t_2) \qquad g_1^{-1}(t_1, t_2) = e^{-t_2}$$

$$g_2(t_1, t_2) = -\log t_1 \qquad g_2^{-1}(t_1, t_2) = e^{-(t_1 - t_2)}$$

and the Jacobian of the transformation is given by

$$J(v, z) = \begin{vmatrix} 0 & -e^{-z} \\ -e^{-(v-z)} & e^{-(v-z)} \end{vmatrix} = e^{-v}$$

and hence

$$f_{V,Z}(v, z) = f_{X,Y}(e^{-z}, e^{-(v-z)})\ e^{-v} = e^{-v} \qquad (v, z) \in \mathbb{V}^{(2)} = \{(v, z) : 0 < z < v < \infty\}$$

and zero otherwise, and so

$$f_V(v) = \int_{-\infty}^{\infty} f_{V,Z}(v, z)\ dz = \int_0^v e^{-v}\ dz = ve^{-v} \qquad v > 0$$

and zero otherwise. Note that we can attempt the joint transformation by setting

$$\begin{array}{l} U = X/Y \\ V = -\log(XY) \end{array} \iff \begin{array}{l} X = U^{1/2} e^{-V/2} \\ Y = U^{-1/2} e^{-V/2} \end{array}$$

note that, as $X$ and $Y$ lie in $(0, 1)$ we have $XY < X/Y$ and $XY < Y/X$, giving constraints $e^{-V} < U$ and $e^{-V} < 1/U$, so that $0 < e^{-V} < \min\{U, 1/U\}$. The Jacobian of the transformation is

$$J(u, v) = \begin{vmatrix} \dfrac{u^{-1/2} e^{-v/2}}{2} & -\dfrac{u^{1/2} e^{-v/2}}{2} \\ -\dfrac{u^{-3/2} e^{-v/2}}{2} & -\dfrac{u^{-1/2} e^{-v/2}}{2} \end{vmatrix} = u^{-1} e^{-v}/2.$$

Hence

$$f_{U,V}(u,v) = u^{-1}e^{-v}/2 \qquad 0 < e^{-v} < \min\{u, 1/u\}, \; u > 0$$

The corresponding marginals are given below: let $g(y) = -\log(\min\{u, 1/u\})$, then

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \int_{g(y)}^{\infty} \frac{e^{-v}}{2u}\, dv = \left[-\frac{e^{-v}}{2u}\right]_{g(y)}^{\infty} = \frac{\min\{u, 1/u\}}{2u} \qquad u > 0$$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, du = \int_{e^{-v}}^{e^{v}} \frac{e^{-v}}{2u}\, du = \left[\frac{\log u}{2} e^{-v}\right]_{e^{-v}}^{e^{v}} = ve^{-v} \qquad v > 0$$

(b) Now let

$$\begin{array}{c} V = X + Y \\ \\ Z = X - Y \end{array} \iff \begin{array}{c} X = \dfrac{V+Z}{2} \\ \\ Y = \dfrac{V-Z}{2} \end{array}$$

and the Jacobian of the transformation is $1/2$. The transformed variables take values on the square $A$ in the $(V, Z)$ plane with corners at $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$ bounded by the lines $z = -v$, $z = 2 - v$, $z = v$ and $z = v - 2$. Then

$$f_{V,Z}(v,z) = \frac{1}{2} \qquad (v,z) \in A$$

and zero otherwise (hint: sketch the square $A$). Hence, integrating in horizontal strips in the $(V, Z)$ plane,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{V,Z}(v,z)\, dv = \begin{cases} \displaystyle\int_{-z}^{2+z} \frac{1}{2}\, dv = 1 + z & -1 < z \leq 0 \\ \\ \displaystyle\int_{z}^{2-z} \frac{1}{2}\, dv = 1 - z & 0 < z < 1 \end{cases}$$

3. The transformations are

$$\begin{array}{l} Y_1 = \dfrac{X_1}{X_1 + X_2 + X_3} \\ \\ Y_2 = \dfrac{X_1}{X_1 + X_2 + X_3} \\ \\ Y_3 = X_1 + X_2 + X_3 \end{array} \iff \begin{array}{l} X_1 = Y_1 Y_3 \\ \\ X_2 = Y_2 Y_3 \\ \\ X_3 = Y_3(1 - Y_1 - Y_2) \end{array}$$

which gives Jacobian

$$J(y_1, y_2, y_3) = \begin{vmatrix} y_3 & 0 & y_1 \\ 0 & y_3 & y_2 \\ -y_3 & -y_3 & (1 - y_1 - y_2) \end{vmatrix} = y_3^2$$

Hence the joint pdf is given by

$$\begin{aligned} f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) &= f_{X_1,X_2,X_3}(y_1y_3, y_2y_3, y_3(1-y_1-y_2))J(y_1,y_2,y_3) \\ &= c_1y_1y_3\exp\{-y_1y_3\}\ \ c_2y_2^2y_3^2\exp\{-y_2y_3\}\ \ c_3y_3^3(1-y_1-y_2)^3\exp\{-y_3(1-y_1-y_2)\}\ \ y_3^2 \\ &= c_1c_2c_3y_1y_2^2(1-y_1-y_2)^3\ y_3^8\exp\{-y_3\} = f_{Y_1,Y_2}(y_1,y_2)f_{Y_3}(y_3) \end{aligned}$$

where

$$f_{Y_1,Y_2}(y_1,y_2) \propto y_1y_2^2(1-y_1-y_2)^3. \quad \text{and} \quad f_{Y_3}(y_3) \propto y_3^8\exp\{-y_3\}$$

Hence $Y_3 \sim Gamma(9,1)$; the transformations give the constraints $0 < Y_1, Y_2 < 1$ and $0 < Y_1 + Y_2 < 1$, and $Y_3 > 0$. Now

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_2 = \int_0^{1-y_1} cy_1y_2^2(1-y_1-y_2)^3\,dy_2 = cy_1(1-y_1)^6\int_0^1 t^2(1-t)^3\,dt \quad (t = y_2/(1-y_1))$$

and hence

$$f_{Y_1}(y_1) \propto y_1(1-y_1)^6 \Longrightarrow Y_1 \sim Beta(2,7),\ f_{Y_1}(y_1) = 336y_1(1-y_1)^6 \qquad 0 < y_1 < 1$$

and hence

$$\mathrm{E}_{f_{Y_1}}[\,Y_1\,] = \frac{2}{2+7} = \frac{2}{9}$$

as the expectation of a $Beta(\alpha,\beta)$ distribution is $\alpha/(\alpha+\beta)$ from notes.

4. (a) Put $U = X/Y$ and $V = Y$; the inverse transformations are therefore $X = UV$ and $Y = V$. In terms of the multivariate transformation theorem, we have transformation functions defined by

$$g_1(t_1,t_2) = t_1/t_2 \qquad g_1^{-1}(t_1,t_2) = t_1t_2$$

$$g_2(t_1,t_2) = t_2 \qquad g_2^{-1}(t_1,t_2) = t_2$$

and the Jacobian of the transformation is given by

$$|J(u,v)| = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = |v|$$

and hence

$$f_{U,V}(u,v) = f_{X,Y}(uv,v)\ |v| = \left(\frac{1}{2\pi}\right)\exp\left\{-\frac{1}{2}(u^2v^2+v^2\right\}|v| \qquad (u,v)\in\mathbb{R}^2$$

and zero otherwise, and so, for any real $u$,

$$\begin{aligned} f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv &= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\right)\exp\{-\tfrac{1}{2}(u^2v^2+v^2\}\,|v|\ dv \\ &= \left(\frac{1}{\pi}\right)\int_0^{\infty} v\exp\left\{-\frac{v^2}{2}(1+u^2)\right\}\ dv && \text{integrand is even func} \\ &= \left(\frac{1}{\pi}\right)\left[-\frac{1}{(1+u^2)}\exp\left\{-\frac{v^2}{2}(1+u^2)\right\}\right]_0^{\infty} = \tfrac{1}{\pi(1+u^2)} && \text{by direct integration} \end{aligned}$$

(b) Now put $T = X/\sqrt{S/\nu}$ and $R = S$; the inverse transformations are therefore $X = T\sqrt{R/\nu}$ and $S = R$. In terms of the multivariate transformation theorem, we have transformation functions from $(X, S) \to (T, R)$ defined by

$$g_1(t_1, t_2) = t_1/\sqrt{t_2/\nu} \qquad\qquad g_1^{-1}(t_1, t_2) = t_1\sqrt{t_2/\nu}$$

$$g_2(t_1, t_2) = t_2 \qquad\qquad g_2^{-1}(t_1, t_2) = t_2$$

and the Jacobian of the transformation is given by

$$|J(t,r)| = \begin{vmatrix} \sqrt{\dfrac{r}{\nu}} & \dfrac{t}{2\sqrt{r\nu}} \\ 0 & 1 \end{vmatrix} = \left|\sqrt{\frac{r}{\nu}}\right| = \sqrt{\frac{r}{\nu}}$$

and hence

$$f_{T,R}(t,r) = f_{X,S}\left(t\sqrt{\frac{r}{\nu}}, r\right)\sqrt{\frac{r}{\nu}} \quad = \quad f_X\left(t\sqrt{\frac{r}{\nu}}\right) f_S(r)\sqrt{\frac{r}{\nu}} \qquad t \in \mathbb{R}, s \in \mathbb{R}^+$$

and zero otherwise, and so, for any real $t$,

$$\begin{aligned}
f_T(t) &= \int_{-\infty}^{\infty} f_{T,R}(t,r)\, dr \\
&= \int_0^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{rt^2}{2\nu}\right\} c(\nu) r^{\nu/2-1} e^{-r/2}\sqrt{\frac{r}{\nu}}\, dr \\
&= \left(\frac{1}{2\pi}\right)^{1/2} \frac{c(\nu)}{\sqrt{\nu}} \int_0^{\infty} r^{(\nu+1)/2-1} \exp\left\{-\frac{r}{2}\left(1+\frac{t^2}{\nu}\right)\right\} dr \\
&= \left(\frac{1}{2\pi}\right)^{1/2} \frac{c(\nu)}{\sqrt{\nu}} \left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \int_0^{\infty} z^{(\nu+1)/2-1} \exp\left\{-\frac{z}{2}\right\} dz \quad \text{setting } z = r\left(1+\frac{t^2}{\nu}\right) \\
&= \left(\frac{1}{2\pi}\right)^{1/2} \frac{c(\nu)}{\sqrt{\nu}} \left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \frac{1}{c(\nu+1)} \qquad \text{integrand is a pdf}
\end{aligned}$$

We also see/deduce that $f_S$ is a $Gamma(\nu/2, 1/2)$ or $Chiquared(\nu)$ density, and that the normalizing constant $c(\nu)$ is given by

$$c(\nu) = \frac{\left(\dfrac{1}{2}\right)^{\nu/2}}{\Gamma\left(\dfrac{\nu}{2}\right)} \qquad \Longrightarrow \qquad f_T(t) = \frac{\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)} \left(\frac{1}{\pi\nu}\right)^{1/2} \frac{1}{(1+t^2/\nu)^{(\nu+1)/2}}$$

which is the $Student(\nu)$ density.

5. We have

$$f_{X|Y}(x|y) = \sqrt{\frac{y}{2\pi}} \exp\left\{-\frac{yx^2}{2}\right\} \qquad x \in \mathbb{R} \qquad\qquad f_Y(y) = c(\nu) y^{\nu/2-1} e^{-\nu y/2} \qquad y \in \mathbb{R}^+$$

where $\nu$ is a positive integer, so that $X|Y = y \sim N(0, y^{-1})$ and $Y \sim Gamma(\nu/2, \nu/2)$, and the normalizing constant $c(\nu)$ is given by

$$c(\nu) = \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)}$$

Now, by the chain rule

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y) \qquad x \in \mathbb{R}, y \in \mathbb{R}^+$$

and zero otherwise, and so, for any real $x$,

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \\
&= \int_0^{\infty} \sqrt{\frac{y}{2\pi}} \exp\left\{-\frac{yx^2}{2}\right\} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} y^{\nu/2-1} e^{-\nu y/2}\, dy \\
&= \frac{1}{\sqrt{2\pi}} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^{\infty} y^{(\nu+1)/2-1} \exp\left\{-\frac{y}{2}\left(\nu + x^2\right)\right\}\, dy \\
&= \frac{1}{\sqrt{2\pi}} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\left(\frac{1}{2}\left(\nu + x^2\right)\right)^{(\nu+1)/2}} \qquad \text{integrand} \propto \text{a Gamma pdf}
\end{aligned}$$

Therefore $f_X$ is given by

$$f_X(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{1}{\pi\nu}\right)^{1/2} \frac{1}{\left(1 + x^2/\nu\right)^{(\nu+1)/2}}$$

which is again the $Student(\nu)$ density.

Exercises 5 and 6 give the two alternative ways of specifying the Student-t distribution, either as a function of independent Normal and Gamma/Chi-squared variables, or as the marginal obtained by "*scale-mixing*" a Normal distribution by a Gamma distribution (that is, rather than having a fixed variance $\sigma^2 = 1/Y$,.we regard $Y$ as a *random variable* having a Gamma distribution, so that $(X, Y)$ have a joint distribution

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y)$$

from which we calculate $f_X(x)$ by integration.