M2S1 : EXERCISE SHEET 4 : SOLUTIONS

1 (a) To calculate the mgf

$$
\begin{aligned}
M_Z(t) &= E_{f_Z}[\, e^{tZ} \,] = \int_{-\infty}^{\infty} e^{zt} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{z^2}{2} \right\} \, dz = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{(z-t)^2}{2} \right\} \, dz \\
&= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{u^2}{2} \right\} \, du = e^{t^2/2}
\end{aligned}
$$

completing the square in z, and then setting $u = z - t$, as the integrand is a pdf.
Now, using the transformation theorem for univariate, 1-1 transformations we have $X = \mu + \dfrac{1}{\lambda} Z \iff Z = \lambda(X - \mu)$, so

$$
f_X(x) = f_Z(\lambda(x-\mu)) \; \lambda = \frac{\lambda}{\sqrt{2\pi}} \exp\left\{ -\frac{\lambda^2}{2}(x-\mu)^2 \right\} \qquad x \in \mathbb{R}
$$

To calculate the mgf of X, use the expectation result given in lectures

$$
M_X(t) = E_{f_Z}\left[e^{t(\mu + Z/\lambda)} \right] = e^{\mu t} M_Z(t/\lambda) = \exp\left\{ \mu t + \frac{t^2}{2\lambda^2} \right\}
$$

The expectation of X is

$$
\begin{aligned}
E_{f_X}[X] &= \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{-\infty}^{\infty} x \left(\frac{\lambda^2}{2\pi} \right)^{1/2} \exp\left\{ -\frac{\lambda^2}{2}(x-\mu)^2 \right\} \, dx \\
&= \int_{-\infty}^{\infty} \left(\mu + t\lambda^{-1} \right) \left(\frac{\lambda^2}{2\pi} \right)^{1/2} \exp\left\{ -\frac{t^2}{2} \right\} \lambda^{-1} \, dt \qquad t = \lambda(x-\mu) \\
&= \mu \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \right)^{1/2} \exp\left\{ -\frac{t^2}{2} \right\} \, dt + \lambda^{-1} \int_{-\infty}^{\infty} t \left(\frac{1}{2\pi} \right)^{1/2} \exp\left\{ -\frac{t^2}{2} \right\} \, dt \\
&= \mu
\end{aligned}
$$

as the first integral is 1, and the second integral is zero, as the integrand is an ODD function about zero. Hence

$$
E_{f_X}[X] = \mu
$$

and note that it is generally true that if a pdf is symmetric about a particular value, then that value is the expectation (if the expectation integral is finite). Alternately, could use the mgf result that says

$$
E_{f_X}[X] = \frac{d}{ds}\left\{ M_X(s) \right\}_{s=0} = M_X^{(1)}(0)
$$

say, so that

$$
E_{f_X}[X] = \frac{d}{ds}\left\{ \exp\left\{ \mu s + \frac{s^2}{2\lambda^2} \right\} \right\}_{s=0} = \left\{ \left(\mu + \frac{s}{\lambda^2} \right) \exp\left\{ \mu s + \frac{s^2}{2\lambda^2} \right\} \right\}_{s=0} = \mu
$$

The expectation of $g(X) = e^X$ is

$$\begin{aligned}
E_{f_X}[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_{-\infty}^{\infty} e^x \left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda^2}{2}(x-\mu)^2\right\}\, dx \\
&= \int_{-\infty}^{\infty} \exp\left\{\mu + t\lambda^{-1}\right\} \left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\} \lambda^{-1}\, dt \qquad \text{setting } t = \lambda(x-\mu) \\
&= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{\mu + t\lambda^{-1} - \frac{t^2}{2}\right\}\, dt = \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(t^2 - 2t\lambda^{-1} - 2\mu\right)\right\}\, dt
\end{aligned}$$

Completing the square in the exponent, we have

$$\left(t^2 - 2t\lambda^{-1} - 2\mu\right) = \left(t - \lambda^{-1}\right)^2 - \left(2\mu + \lambda^{-2}\right)$$

and hence

$$\begin{aligned}
E_{f_X}[g(X)] &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(t - \lambda^{-1}\right)^2 + \left(\mu + \frac{1}{2\lambda^2}\right)\right\}\, dt \\
&= \exp\left\{\mu + \frac{1}{2\lambda^2}\right\} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(t - \lambda^{-1}\right)^2\right\}\, dt = \exp\left\{\mu + \frac{1}{2\lambda^2}\right\}
\end{aligned}$$

as the integral is equal to 1, as it is the integral of a pdf for all choices of λ.

(b) If $Y = e^X$, so $\mathbb{Y} = \mathbb{R}^+$, and from first principles we have

$$F_Y(y) = \mathrm{P}[Y \le y] = \mathrm{P}\left[e^X \le y\right] = \mathrm{P}[X \le \log y] = F_X(\log y) \qquad \Longrightarrow \qquad f_Y(y) = f_X(\log y)\,\frac{1}{y} \qquad y > 0$$

Note that the function $g(t) = e^t$ is a monotone increasing function, with $g^{-1}(t) = \log t$, so that we can use the general result directly, that is

$$f_Y(y) = f_X(g^{-1}(y))\ J(y) \qquad \text{where} \qquad J(y) = \left|\frac{d}{dt}\left\{g^{-1}(t)\right\}_{t=y}\right| = \left|\frac{d}{dt}\left\{\log t\right\}_{t=y}\right| = \frac{1}{y}$$

Hence

$$f_Y(y) = \frac{1}{y}\left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda^2}{2}(\log y - \mu)^2\right\} \qquad y > 0.$$

For the expectation, we have from first principles

$$\begin{aligned}
E_{f_Y}[Y] &= \int_0^{\infty} y f_Y(y)\, dy = \int_{-\infty}^{\infty} y\,\frac{1}{y}\left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda}{2}(\log y - \mu)^2\right\}\, dy \\
&= \int_{-\infty}^{\infty} \left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda^2}{2}(t-\mu)^2\right\} e^t\, dt \quad = \quad \exp\left\{\mu + \frac{1}{2\lambda^2}\right\}
\end{aligned}$$

where $t = \log y$, as the integral is precisely the one carried out above. This illustrates the transformation/expectation result that, if $Y = g(X)$, then

$$E_{f_Y}[Y] = E_{f_X}[g(X)]$$

(c) If $T = Z^2$, then from first principles

$$F_T(t) = P[\ T \leq t\ ] = P[\ Z^2 \leq t\ ] = P[\ -\sqrt{t} \leq Z \leq \sqrt{t}\ ]$$

$$\Longrightarrow f_T(t) = \frac{1}{2\sqrt{t}}\left[f_Z(\sqrt{t}) + f_Z(-\sqrt{t})\right] = \frac{1}{\sqrt{2\pi}}t^{-1/2}\exp\left\{-\frac{t}{2}\right\} \qquad t > 0$$

and hence

$$M_T(t) = E_{f_T}[\ e^{tT}\ ] = \int_{-\infty}^{\infty} e^{tx} f_T(x)\ dx = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi x}}\exp\left\{-\frac{x}{2}\right\}\ dx = \int_0^{\infty}\frac{1}{\sqrt{2\pi x}}\exp\left\{-\frac{(1-2t)x}{2}\right\}\ dx$$

$$= \left(\frac{1}{1-2t}\right)^{1/2}\int_0^{\infty}\frac{1}{\sqrt{2\pi y}}\exp\left\{-\frac{y}{2}\right\}\ dy = \left(\frac{1}{1-2t}\right)^{1/2}$$

where $y = (1-2t)x$, as the integrand is a pdf.

2. By definition of mgfs for discrete variables, we can deduce immediately that, as

$$M_X(t) = \sum_{x=-\infty}^{\infty} e^{tx} f_X(x)$$

$P[\ X = x\ ]$ is just the coefficient of e^{tx} in the expression for M_X, and hence $P[\ X = 1\ ] = 1/8$, $P[\ X = 2\ ] = 1/4$ and $P[\ X = 3\ ] = 5/8$. Also, we have $E_{f_X}[\ X^r\ ] = M_X^{(r)}(0)$, so that

$$E_{f_X}[\ X\ ] = M_X^{(1)}(0) = \frac{1}{8} + 2\frac{1}{4} + 3\frac{5}{8} = \frac{5}{2} \qquad E_{f_X}[\ X^2\ ] = M_X^{(2)}(0) = \frac{1}{8} + 4\frac{1}{4} + 9\frac{5}{8} = \frac{27}{4}$$

$$\Longrightarrow Var_{f_X}[\ X\ ] = E_{f_X}[\ X^2\ ] - \{E_{f_X}[\ X\ ]\}^2 = \frac{1}{2}$$

3. Can identify that $X \sim Binomial(n, \theta)$, but in any case,

$$M_X(t) = (1-\theta+\theta e^t)^n = (1+(e^t-1)\theta)^n = \left(1+\theta\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+...\right)\right)^n$$

$$= \sum_{r=0}^{n}\binom{n}{r}\theta^r\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+...\right)^r$$

and from the mgf definition $E_{f_X}[\ X^r\ ]$ is $r!$ times the coefficient of t^r. Difficult to identify this general term, but can easily identify the coefficient of t as $n\theta = E_{f_X}[\ X\ ]$, and the coefficient of t^2 as $n\theta + n(n-1)\theta^2 = E_{f_X}[\ X^2\ ]$ etc.

4. For this pdf,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\ dx = \int_{-2}^{\infty} e^{tx} e^{-(x+2)}\ dx = e^{-2}\int_{-2}^{\infty} e^{-(1-t)x}\ dx$$

$$= \frac{e^{-2}}{1-t}\int_{-2(1-t)}^{\infty} e^{-y}\ dy = \frac{e^{-2}}{1-t}\ \left[-e^{-y}\right]_{-2(1-t)}^{\infty} = \frac{e^{-2t}}{1-t} \qquad t < 1$$

Now

$$M_X^{(1)}(t) = \frac{e^{-2t}}{(1-t)^2}(2t-1) \qquad M_X^{(2)}(t) = \frac{e^{-2t}}{(1-t)^3}\left[1+(2t-1)^2\right]$$

so that $M_X^{(1)}(0) = -1 = E_{f_X}[\ X\ ]$ and $M_X^{(2)}(0) = 2 = E_{f_X}[\ X^2\ ] \Longrightarrow \mathrm{Var}_{f_X}[\ X\ ] = 1$

5. We have $K_X(t) = \log M_X(t)$, hence

$$K_X^{(1)}(t) = \frac{d}{ds}\{K_X(t)\}_{s=t} = \frac{d}{ds}\{\log M_X(t)\}_{s=t} = \frac{M_X^{(1)}(t)}{M_X(t)} \Longrightarrow K_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = E_{f_X}[\ X\ ]$$

as $M_X(0) = 1$. Similarly

$$K_X^{(2)}(t) = \frac{M_X(t)M_X^{(2)}(t) - \left\{M_X^{(1)}(t)\right\}^2}{\{M_X(t)\}^2}$$

and hence

$$K_X^{(2)}(0) = \frac{M_X(0)M_X^{(2)}(0) - \left\{M_X^{(1)}(0)\right\}^2}{\{M_X(0)\}^2} = E_{f_X}[\ X^2\ ] - \{E_{f_X}[\ X\ ]\}^2$$

and hence $K_X^{(2)}(0) = Var_{f_X}[\ X\ ]$

6. Easy to see that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, with $\mathbb{X}^{(2)} = \mathbb{X} \times \mathbb{Y}$, so X and Y are independent, where

$$f_X(x) = \sqrt{c}\,\frac{2^x}{x!} \qquad f_Y(y) = \sqrt{c}\,\frac{2^y}{y!} \qquad \text{and} \qquad \sum_{x=0}^{\infty} f_X(x) = 1 \Longrightarrow \sqrt{c} = e^{-2}$$

(marginal mass functions must have identical forms as joint mass function is symmetric in x and y) as the summation is identical to the power series expansion of e^z at $z = 2$ if $\sqrt{c} = e^{-2}$.

7. $F_{X,Y}$ is continuous and non decreasing in x and y, and

$$\lim_{x \longrightarrow -\infty} F_{X,Y}(x,y) = \lim_{y \longrightarrow -\infty} F_{X,Y}(x,y) = 0 \qquad \lim_{x,y \longrightarrow \infty} F_{X,Y}(x,y) = 1$$

so $F_{X,Y}$ is a valid cdf, and

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial t_1 \partial t_2}\{F_{X,Y}(t_1,t_2)\}_{t_1=x,t_2=y} = \frac{e^{-x}}{\pi(1+y^2)} = f_X(x)f_Y(y)$$

so as $\mathbb{X}^{(2)} = \mathbb{R}^+ \times \mathbb{R}$, X and Y are independent.

8. (i) If $\mathbb{X}^{(2)}=(0,1) \times (0,1)$ is the (joint) range of vector random variable (X,Y). We have

$$f_{X,Y}(x,y) = cx(1-y) \qquad 0 < x < 1,\ 0 < y < 1$$

so that

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \qquad \text{and} \qquad \mathbb{X}^{(2)} = \mathbb{X} \times \mathbb{Y}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are the ranges of X and Y respectively, and

$$f_X(x) = c_1 x \qquad \text{and} \qquad f_Y(y) = c_2(1-y) \tag{1}$$

for some constants satisfying $c_1 c_2 = c$. Hence, the two conditions for independence are satisfied in (2), and X and Y are independent.

(ii) We must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dxdy = 1 \therefore c^{-1} = \int_{0}^{1}\int_{0}^{1} x(1-y)\ dxdy = 1$$

and as

$$\int_{0}^{1}\int_{0}^{1} x(1-y)\ dxdy = \left\{\int_{0}^{1} x\ dx\right\}\left\{\int_{0}^{1}(1-y)\ dy\right\} = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$$

we have $c = 4$.

(iii) We have $A = \{(x,y)\ :\ 0 < x < y < 1\}$, and hence, recalling that the joint density is only non-zero when $x < y$, we first fix a y and integrate dx on the range $(0,y)$, and then integrate dy on the range $(0,1)$, that is

$$\begin{aligned} P[\ X < Y\ ] &= \int_A\int f_{X,Y}(x,y)\ dxdy = \int_0^1\left\{\int_0^y 4x(1-y)\ dx\right\} dy = \int_0^1\left\{\int_0^y x\ dx\right\} 4(1-y)\ dy \\ &= \int_0^1 2y^2(1-y)\ dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^4\right]_0^1 = \frac{1}{6} \end{aligned}$$

9. The joint pdf of X and Y is given by

$$f_{X.Y}(x,y) = 24xy \qquad x > 0,\ y > 0,\ x + y < 1$$

and zero otherwise, the marginal pdf f_X is given by

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^{1-x} 24xy\ dy = 24x\left[\frac{y^2}{2}\right]_0^{1-x} \\ &= 12x(1-x)^2 \qquad\qquad 0 < x < 1 \end{aligned}$$

as the integrand is only non-zero when $0 < x + y < 1 \Longrightarrow 0 < y < 1 - x$ for fixed x

10. First sketch the support of the density; this will make it clear that the boundaries of the support are different for $0 < y \leq 1$ and $y > 1$. The marginals are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_{1/x}^{x} \frac{1}{2x^2 y}\ dy = \frac{1}{2x^2}(\log x - \log(1/x)) = \frac{\log x}{x^2} \qquad 1 \leq x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \begin{cases} \displaystyle\int_{1/y}^{\infty} \frac{1}{2x^2 y}\ dx = \frac{1}{2} & 0 \leq y \leq 1 \\ \displaystyle\int_{y}^{\infty} \frac{1}{2x^2 y}\ dx = \frac{1}{2y^2} & 1 \leq y \end{cases}$$

Conditionals:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{x^2 y} & 1/y \leq x \text{ if } 0 \leq y \leq 1 \\ \dfrac{y}{x^2} & y \leq x \text{ if } 1 \leq y \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{2y \log x} \qquad 1/x \leq y \leq x \text{ if } x \geq 1$$

Marginal expectation of Y;

$$E_{f_Y}[\,Y\,] = \int_{-\infty}^{\infty} y f_Y(y)\ dy = \int_0^1 \frac{y}{2}\ dy + \int_1^{\infty} \frac{1}{2y}\ dy = \infty$$

as the second integral is divergent.