M2S1 : ASSESSED COURSEWORK 3 : SOLUTIONS

1. (a) Using the results given in lectures:If

$$Y = AX \qquad \text{where} \qquad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}.$$

and $X \sim N(\mathbf{0}, I_3)$, where $\mathbf{0}$ is the zero vector, and I_3 is the 3×3 identity matrix, then

$$Y \sim N\left(A\mathbf{0}, AI_3A^T\right) \equiv N(\mathbf{0}, \Sigma)$$

where

$$\Sigma = AA^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 4 \\ -1 & 5 & -4 \\ 4 & -4 & 16 \end{pmatrix}$$

[4 MARKS]

(b) The covariance between Y_1 and Y_3 is given by Σ_{13}, that is, $Cov_{f_{Y_1,Y_3}}[Y_1, Y_3] = 4$.

[4 MARKS]

(c) By the transformation result from (a), the marginal distribution of Y_1 is obtained by the result

$$Y_1 = BY \sim N\left(B\mathbf{0}, B\Sigma B^T\right)$$

where $B\Sigma B^T = \Sigma_{11} = 2$ and hence

$$Y_1 \sim N(0,2)$$

(can deduce this directly, using MGFs, and the definition of $Y_1 = X_1 + X_3$).

[2 MARKS]

(d) We have for the mgf for X, by using the mgf of the standard normal $\exp\{t^2/2\}$

$$\begin{aligned} M_X(\mathbf{t}) &= E_{f_X}\left[\exp\left\{\mathbf{t}^T X\right\}\right] = E_{f_X}\left[\exp\left\{t_1X_1 + t_2X_2 + t_3X_3\right\}\right] = \prod_{i=1}^{3} \exp\left\{t_iX_i\right\} \\ &\qquad\qquad \text{(by independence)} \\ &= \prod_{i=1}^{3} M_{X_i}(t_i) = \exp\left\{\frac{t_1^2}{2}\right\}\exp\left\{\frac{t_2^2}{2}\right\}\exp\left\{\frac{t_3^2}{2}\right\} = \exp\left\{\frac{\mathbf{t}^T\mathbf{t}}{2}\right\} \\ M_Y(\mathbf{t}) &= E_{f_Y}\left[\exp\left\{\mathbf{t}^T Y\right\}\right] = E_{f_X}\left[\exp\left\{\mathbf{t}^T(AX)\right\}\right] \\ &= E_{f_X}\left[\exp\left\{\mathbf{t}^T\left(A^T\right)^T X\right\}\right] = E_{f_X}\left[\exp\left\{\left(A^T\mathbf{t}\right)^T X\right\}\right] \\ &= M_X\left(\left(A^T\mathbf{t}\right)\right) = \exp\left\{\frac{\left(A^T\mathbf{t}\right)^T\left(A^T\mathbf{t}\right)}{2}\right\} = \exp\left\{\frac{\mathbf{t}^T\left(AA^T\right)\mathbf{t}}{2}\right\} = \exp\left\{\frac{\mathbf{t}^T\Sigma\mathbf{t}}{2}\right\} \end{aligned}$$

[2 MARKS, 3 MARKS]

(e) We have

$$\begin{aligned} V &= Y^T\Sigma^{-1}Y = (AX)^T \left(AA^T\right)^{-1} AX = X^T A^T \left(A^T\right)^{-1} A^{-1}AX = X^T \left(A^T \left(A^T\right)^{-1}\right) \left(A^{-1}A\right) X \\ &= X^T X = X_1^2 + X_2^2 + X_3^2. \end{aligned}$$

Now, X_1, X_2 and X_3 are all $Normal(0,1)$, and thus for $i = 1,2,3$, from lecture notes

$$V_i = X_i^2 \sim \chi_1^2 \equiv Gamma\left(\frac{1}{2},\frac{1}{2}\right) \qquad \therefore \qquad V = \sum_{i=1}^{3} V_i \sim Gamma\left(\frac{3}{2},\frac{1}{2}\right) \equiv \chi_3^2$$

using mgfs, or the result for the addition of independent Gamma random variables with the same β parameter.

[5 MARKS]

2. (a) We have for $y > 0$

$$F_{Y_n}(y) = \{F_X(y)\}^n = \left\{\frac{(y+1)^2 - 1}{(y+1)^2}\right\}^n = \left\{1 - \frac{1}{(y+1)^2}\right\}^n$$

with $F_{Y_n}(y) = 0$ for $y \leq 0$.

(i) For any (fixed) $y > 0$, $0 < (y+1)^{-2} < 1$, and hence

$$\left\{1 - \frac{1}{(y+1)^2}\right\}^n \to 0 \qquad \text{as } n \to \infty \qquad \therefore F_{Y_n}(y) \to F(y) = 0$$

for all $y \in \mathbb{R}$, which is **not** a probability distribution function, so the limiting distribution **does not exist**.

[5 MARKS]

(ii) For the transformed variable $Z_n = Y_n/\sqrt{n}$, for $z > 0$,

$$F_{Z_n}(z) = P\left[Z_n \leq z\right] = P\left[Y_n/\sqrt{n} \leq z\right] = P\left[Y_n \leq \sqrt{n}z\right] = F_{Y_n}\left(\sqrt{n}z\right).$$

$$\therefore F_{Z_n}(z) = \left\{1 - \frac{1}{\left(\sqrt{n}z+1\right)^2}\right\}^n = \left\{1 - \frac{1}{\left(nz^2 + 2\sqrt{n}z + 1\right)}\right\}^n = \left\{1 - \frac{1}{n}\frac{1}{\left(z^2 + 2z/\sqrt{n} + 1/n\right)}\right\}^n$$

In the limit as $n \to \infty$, terms in the denominator of the bracketed expression tend to zero with n at a fast enough rate to preserve the result that

$$\lim_{n\to\infty}\left\{1 - \frac{1}{n}\frac{1}{\left(z^2 + 2z/\sqrt{n} + 1/n\right)}\right\}^n = \lim_{n\to\infty}\left\{1 - \frac{1}{nz^2}\right\}^n = e^{-1/z^2}$$

as $n \to \infty$, as in the Central Limit Theorem proof from lectures. Thus, for $z > 0$

$$F_{Z_n}(z) \to F_Z(z) = \exp\left\{-\frac{1}{z^2}\right\} \qquad (\text{and } F_{Z_n}(z) \to 0 \text{ for } z \leq 0)$$

which is a valid cdf, the limiting distribution exists (and, in fact, is continuous).

[5 MARKS]

(b) Now suppose that $X_1, ...X_n$ are independent $Exponential(1)$ random variables.

(i) We have

$$F_{Y_n}(y) = \{F_X(y)\}^n = \left\{1 - e^{-y}\right\}^n \qquad \text{for } z > 0$$

and is zero for $z \leq 0$.

[2 MARKS]

(ii) For $Z_n = Y_n - a$, we have that the range of Z_n is $(-a, \infty)$

$$F_{Z_n}(z) = P[Z_n \leq z] = P[Y_n - a \leq z] = P[Y_n \leq z + a] = F_{Y_n}(z + a)$$

and hence

$$F_{Z_n}(z) = \left\{1 - e^{-(z+a)}\right\}^n$$

for $z > -a$, and $F_{Z_n}(z) = 0$ for $z \leq -a$.

[2 MARKS]

(iii) For constants $\{a_n\}$

$$F_{Z_n}(z) = \left\{1 - e^{-(z+a_n)}\right\}^n = \left\{1 - e^{-a_n} e^{-z}\right\}^n = \left\{1 - A_n e^{-z}\right\}^n$$

where $A_n = e^{-a_n}$. Now, if $A_n = 1/n$ (so that $a_n = \log n$), we have, for $z > 0$,

$$F_{Z_n}(z) = \left\{1 - \frac{e^{-z}}{n}\right\}^n \to \exp\left\{-e^{-z}\right\}$$

as $n \to \infty$. This is a valid cdf for a random variable Z on $\mathbb{R}$. Hence, for large n

$$F_{Z_n}(z) = P[Z_n \leq z] \approx \exp\left\{-e^{-z}\right\}$$

and therefore

$$P[Y_n \leq y_0] = P[Z_n - a_n \leq y_0] = P[Z_n \leq y_0 + a_n] = F_{Z_n}(y_0 + a_n) \approx \exp\{-\exp\{-(y_0 + a_n)\}\}$$

so that

$$P[Y_n > y_0] \approx 1 - \exp\{-\exp\{-(y_0 + a_n)\}\}.$$

[2 MARKS]

(iv) Here we have the maximum **observed** between-earthquake time to be 1021 days. Let $X_1, ..., X_n$ ($n = 199$) be the random between-earthquake times and Y_n be the (random) maximum value. We have from above that

$$F_{Y_n}(y) = \{F_X(y)\}^n = \left\{1 - e^{-\lambda y}\right\}^n \qquad \therefore \qquad P[Y_n > y] = 1 - \left\{1 - e^{-\lambda y}\right\}^n$$

In this case, $\lambda = 0.01$, so

$$P[Y_n > y] = 1 - \left\{1 - e^{-0.01y}\right\}^n.$$

But we have **observed** $Y_n = 1021$, and under this model

$$P[Y_n > 1021] = 1 - \left\{1 - e^{-0.01 \times 1021}\right\}^{199} = 0.0073$$

which is an unlikely outcome, given the proposed model. Hence it is probable that the proposed model is incorrect.

Re-doing the calculation with $n = 200$ (this is acceptable, the question was a little ambiguous), we have again $P[Y_n > 1021] = 0.0073$, so there is no significant difference in the conclusion.

[4 MARKS]