M2S1 : ASSESSED COURSEWORK 2 : SOLUTIONS

1. (a) We need

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$$

Now

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = \int_0^1\int_0^y c(1-y)dxdy = c\int_0^1 (1-y)ydy = c\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1 = \frac{c}{6}$$

so therefore $c = 6$.

[2 MARKS]

(b) The marginal for X is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_x^1 6(1-y)dy = 6\left[y-\frac{y^2}{2}\right]_x^1 = 6\left[\frac{1}{2}-\left(x-\frac{x^2}{2}\right)\right] = 3-6x+3x^2 \\ &= 3(1-x)^2 \qquad 0<x<1, \text{ zero otherwise} \end{aligned}$$

[1 MARK]

Hence

$$\begin{aligned} E_{f_X}[X] &= \int_{-\infty}^{\infty} xf_X(x)dx = \int_0^1 x3(1-x)^2 dx = \int_0^1 \left(3x-6x^2+3x^3\right) dx \\ &= \left[\frac{3x^2}{2}-2x^3+\frac{3x^4}{4}\right]_0^1 = \frac{3}{2}-2+\frac{3}{4} = \frac{1}{4} \end{aligned}$$

[1 MARK]

and

$$\begin{aligned} E_{f_X}\left[X^2\right] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^1 x^2 3(1-x)^2 dx = \int_0^1 \left(3x^2-6x^3+3x^4\right) dx \\ &= \left[x^3-\frac{3x^4}{2}+\frac{3x^5}{5}\right]_0^1 = 1-\frac{3}{2}+\frac{3}{5} = \frac{1}{10} \end{aligned}$$

and so

$$Var_{f_X}[X] = E_{f_X}\left[X^2\right] - \{E_{f_X}[X]\}^2 = \frac{1}{10}-\left(\frac{1}{4}\right)^2 = \frac{3}{80}$$

[1 MARK]

(c) By definition, for fixed x with $0<x<1$,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{6(1-y)}{3(1-x)^2} = \frac{2(1-y)}{(1-x)^2} \qquad x<y<1$$

and zero otherwise. Hence, the expected value of $1-Y$ with respect to this conditional distribution for $X=x$ is

$$\begin{aligned} E_{f_{Y|X}}\left[(1-Y)\,|X=x\right] &= \int_{-\infty}^{\infty}(1-y)\,f_{Y|X}(y|x)dy = \int_x^1 (1-y)\frac{2(1-y)}{(1-x)^2}dy = \int_x^1 \frac{2(1-y)^2}{(1-x)^2}dy \\ &= \frac{2}{(1-x)^2}\left[-\frac{1}{3}(1-y)^3\right]_x^1 = \frac{2}{3}(1-x) \end{aligned}$$

[1 MARK]

Therefore

$$E_{f_{Y|X}}[Y|X=x] = 1 - \frac{2}{3}(1-x) = \frac{1}{3} + \frac{2}{3}x$$

and using the Law of Iterated Expectation, and the linearity of expectations

$$E_{f_Y}[Y] = E_{f_X}\left[E_{f_{Y|X}}[Y|X=x]\right] = E_{f_X}\left[\frac{1}{3} + \frac{2}{3}X\right] = \frac{1}{3} + \frac{2}{3}E_{f_X}[X] = \frac{1}{3} + \frac{2}{3} \times \frac{1}{4} = \frac{1}{2}$$

and

$$E_{f_{X,Y}}[XY] = E_{f_X}\left[E_{f_{Y|X}}[XY|X=x]\right] = E_{f_X}\left[X E_{f_{Y|X}}[Y|X=x]\right]$$

as x is a constant in $E_{f_{Y|X}}[XY|X=x]$, so can be taken outside of the integral. Hence

$$E_{f_{X,Y}}[XY] = E_{f_X}\left[X\left(\frac{1}{3} + \frac{2}{3}X\right)\right] = \frac{1}{3}E_{f_X}[X] + \frac{2}{3}E_{f_X}\left[X^2\right] = \frac{1}{3} \times \frac{1}{4} + \frac{2}{3} \times \frac{1}{10} = \frac{3}{20}$$

so that

$$Cov_{f_{X,Y}}[X,Y] = E_{f_{X,Y}}[XY] - E_{f_X}[X]\,E_{f_Y}[Y] = \frac{3}{20} - \frac{1}{4} \times \frac{1}{2} = \frac{1}{40}$$

[2 MARKS]

(d) We have

$$P[Y < 2X] = \int\int_A f_{X,Y}(x,y)\,dxdy$$

where $A \equiv \{(x,y) : 0 < x < y < 1 \text{ and } y < 2x\}$. Thus

$$P[Y < 2X] = \int_0^1 \left\{\int_{y/2}^{y} 6(1-y)\,dx\right\} dy = \int_0^1 3y(1-y)\,dy = \left[\frac{3y^2}{2} - y^3\right]_0^1 = \frac{1}{2}$$

[2 MARKS]

2. (a)(i) U and V are independent, as for all possible pairs $(u,v) \in \mathbb{R}^2$

$$f_{U,V}(u,v) = f_U(u) f_V(v)$$

We do not have to compute f_U and f_V explicitly; we can deduce the result by inspection of the joint pdf, and by noticing that the ranges of the two variables do not interfere with each other, that is, the joint range is a Cartesian product of the range of U and the range of V

[2 MARKS]

(ii) If $A \equiv \{(u,v) : 0 < u < 1, 0 < v < u\}$, we have

$$\begin{aligned}
P[(U,V) \in A] &= \int\int_A f_{U,V}(u,v)\,dudv = \int_0^1 \left\{\int_0^u \frac{3}{2}u^2(1-|v|)\,dv\right\} du \\
&= \int_0^1 \frac{3}{2}u^2 \left\{\int_0^u (1-v)\,dv\right\} du = \int_0^1 \frac{3}{2}u^2\left[v - \frac{v^2}{2}\right]_0^u du = \int_0^1 \frac{3}{2}u^2\left(u - \frac{u^2}{2}\right) du \\
&= \frac{3}{2}\left[\frac{u^4}{4} - \frac{u^5}{10}\right]_0^1 = \frac{3}{2}\left(\frac{1}{4} - \frac{1}{10}\right) = \frac{9}{40}
\end{aligned}$$

[3 MARKS]

(b) For this model

$$P\left[R^2 < S < R\right] = \int_0^1 \left\{\int_{r^2}^{r} 2r\,ds\right\} dr = \int_0^1 2r\left(r - r^2\right) dr = 2\left[\frac{r^3}{3} - \frac{r^4}{4}\right]_0^1 = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

[5 MARKS]