

WORKED EXAMPLES 3

COVARIANCE CALCULATIONS

EXAMPLE Let X and Y be discrete random variables with joint mass function defined by

$$f_{X,Y}(x, y) = \frac{1}{4} \quad (x, y) \in \{(0, 0), (1, 1), (1, -1), (2, 0)\}$$

and zero otherwise. The marginal mass functions, expectations and variances of X and Y are

$$f_X(x) = \sum_y f_{X,Y}(x, y) = \begin{cases} \frac{1}{4} & x = 0, 2 \\ \frac{1}{2} & x = 1 \end{cases}$$

$$\Rightarrow E_{f_X}[X] = \sum_{x=0}^2 x f_X(x) = \left(0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4}\right) = 1$$

$$E_{f_X}[X^2] = \sum_{x=0}^2 x^2 f_X(x) = \left(0 \times \frac{1}{4} + 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4}\right) = \frac{3}{2}$$

$$\Rightarrow \text{Var}_{f_X}[X] = E_{f_X}[X^2] - \{E_{f_X}[X]\}^2 = \frac{3}{2} - \{1\}^2 = \frac{1}{2}$$

$$f_Y(y) = \sum_x f_{X,Y}(x, y) = \begin{cases} \frac{1}{4} & y = -1, 1 \\ \frac{1}{2} & y = 0 \end{cases}$$

$$\Rightarrow E_{f_Y}[Y] = \sum_{y=0}^2 y f_Y(y) = \left(-1 \times \frac{1}{4} + 0 \times \frac{1}{2} + 1 \times \frac{1}{4}\right) = 0$$

$$E_{f_Y}[Y^2] = \sum_{y=0}^2 y^2 f_Y(y) = \left((-1)^2 \times \frac{1}{4} + 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{4}\right) = \frac{1}{2}$$

$$\Rightarrow \text{Var}_{f_Y}[Y] = E_{f_Y}[Y^2] - \{E_{f_Y}[Y]\}^2 = \frac{1}{2} - \{0\}^2 = \frac{1}{2}$$

and to compute the covariance we also need to compute $E_{f_{X,Y}}[XY]$

$$E_{f_{X,Y}}[XY] = \sum_x \sum_y xy f_{X,Y}(x, y) = \left((0 \times 0) \times \frac{1}{4} + (1 \times 1) \times \frac{1}{4} + (1 \times -1) \times \frac{1}{4} + (2 \times 0) \times \frac{1}{4}\right) = 0$$

$$\Rightarrow \text{Cov}_{f_{X,Y}}[X, Y] = E_{f_{X,Y}}[XY] - E_{f_X}[X] E_{f_Y}[Y] = 0 - 1 \times 0 = 0 \quad \text{Corr}_{f_{X,Y}}[X, Y] = 0$$

Hence the two variables have covariance and correlation zero. But note that X and Y are **not independent** as it is **not** true that

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

for **all** x and y .

EXAMPLE Let X and Y be continuous random variables with joint pdf

$$f_{X,Y}(x, y) = 3x \quad 0 \leq y \leq x \leq 1$$

and zero otherwise.

The marginal pdfs, expectations and variances of X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy = \int_0^x 3x dy = 3x^2 \quad 0 \leq x \leq 1$$

$$\Rightarrow E_{f_X}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x \times 3x^2 dx = \left[\frac{3}{4} x^4 \right]_0^1 = \frac{3}{4}$$

$$E_{f_X}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^1 x^2 \times 3x^2 dx = \left[\frac{3}{5} x^5 \right]_0^1 = \frac{3}{5}$$

$$\Rightarrow \text{Var}_{f_X}[X] = E_{f_X}[X^2] - \{E_{f_X}[X]\}^2 = \frac{3}{5} - \left\{ \frac{3}{4} \right\}^2 = \frac{3}{80}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx = \int_y^1 3x dx = \left[\frac{3}{2} x^2 \right]_y^1 = \frac{3}{2}(1 - y^2) \quad 0 \leq y \leq 1$$

$$\Rightarrow E_{f_Y}[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 y \times \frac{3}{2}(1 - y^2) dy = \left[\frac{3}{2} \left(\frac{y^2}{2} - \frac{y^4}{4} \right) \right]_0^1 = \frac{3}{2} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{3}{8}$$

$$E_{f_Y}[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y) dy = \int_0^1 y^2 \times \frac{3}{2}(1 - y^2) dy = \left[\frac{3}{2} \left(\frac{y^3}{3} - \frac{y^5}{5} \right) \right]_0^1 = \frac{3}{2} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{1}{5}$$

$$\Rightarrow \text{Var}_{f_Y}[Y] = E_{f_Y}[Y^2] - \{E_{f_Y}[Y]\}^2 = \frac{1}{5} - \left\{ \frac{3}{8} \right\}^2 = \frac{19}{320}$$

and to compute the covariance we also need to compute $E_{f_{X,Y}}[XY]$

$$\begin{aligned} E_{f_{X,Y}}[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x, y) dy dx = \int_0^1 \int_0^x xy \times 3x dy dx \\ &= \int_0^1 \left\{ \int_0^x y dy \right\} 3x^2 dx = \int_0^1 \left[\frac{y^2}{2} \right]_0^x 3x^2 dx = \int_0^1 \frac{x^2}{2} \times 3x^2 dx \\ &= \frac{3}{2} \left[\frac{x^5}{5} \right]_0^1 = \frac{3}{10} \end{aligned}$$

$$\Rightarrow \text{Cov}_{f_{X,Y}}[X, Y] = E_{f_{X,Y}}[XY] - E_{f_X}[X] E_{f_Y}[Y] = \frac{3}{10} - \frac{3}{4} \times \frac{3}{8} = \frac{3}{160}$$

$$\text{Corr}_{f_{X,Y}}[X, Y] = \frac{\text{Cov}_{f_{X,Y}}[X, Y]}{\sqrt{\text{Var}_{f_X}[X] \times \text{Var}_{f_Y}[Y]}} = \frac{\frac{3}{160}}{\sqrt{\frac{3}{80} \times \frac{19}{320}}} = 0.397$$