M2S1 - EXERCISES 4

GENERATING FUNCTIONS AND TRANSFORMATIONS

1. *Transformations of Normal random variables.*
   The continuous random variable Z with range $\mathbb{Z} = \mathbb{R}$ has pdf given by

$$f_z(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\} \qquad z \in \mathbb{R}$$

(a) Find the mgf of random variable Z, the pdf and the mgf of random variable X where

$$X = \mu + \frac{1}{\lambda}Z.$$

for parameters μ and $\lambda > 0$.

Find the expectation of X, and the expectation of the function $g(X)$ where $g(x) = e^x$.

(b) Suppose now Y is the random variable defined in terms of X by $Y = e^X$ Find the pdf of Y, and show that the expectation of Y is

$$\exp\left\{\mu + \frac{1}{2\lambda^2}\right\}$$

(c) Finally, let random variable T be defined by $T = Z^2$. Find the pdf and mgf of Z.

2. Suppose that random variable X has mgf M_X given by

$$M_X(t) = \frac{1}{8}e^t + \frac{2}{8}e^{2t} + \frac{5}{8}e^{3t}$$

Find the probability distribution, and the expectation and variance of X (*hint: consider G_X, and its definition).*

3. Suppose that random variable X has mgf given by

$$M_X(t) = \left(1 - \theta + \theta e^t\right)^n$$

for some θ, where $0 \leq \theta \leq 1$. Obtain a power series expansion for $M_X(t)$, and hence identify $\mathrm{E}_{f_X}[X^r]$ for $r = 1, 2, 3, ...$

4. Suppose that X is a continuous random variable with pdf

$$f_X(x) = \exp\{-(x+2)\} \qquad -2 < x < \infty$$

Find the mgf of X, and **hence** find the expectation and variance of X.

5. Suppose that X is a random variable with mass function/pdf f_X and mgf M_X. The *cumulant generating function* of X, K_X, is defined by $K_X(t) = \log[M_X(t)]$. Prove that

$$\frac{d}{dt}\{K_X(t)\}_{t=0} = E_{f_X}[X] \qquad \frac{d^2}{dt^2}\{K_X(t)\}_{t=0} = Var_{f_X}[X]$$

DISCRETE AND CONTINUOUS MULTIVARIATE DISTRIBUTIONS

6. Suppose that X and Y are discrete random variables with joint mass function given by

$$f_{X,Y}(x,y) = c\frac{2^{x+y}}{x!y!} \qquad x,\ y = 0,1,2,....$$

and zero otherwise, for some constant c. Find the value of c, and the marginal mass functions of X and Y. Prove that X and Y are independent random variables.

7. Continuous random variables X and Y have joint cdf, $F_{X,Y}$ defined for $(x,y) \in \mathbb{R}^2$ by

$$F_{X,Y}(x,y) = \left(1 - e^{-x}\right)\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1} y\right) \qquad x \geq 0$$

and zero otherwise. Find the joint pdf, $f_{X,Y}$. Are X and Y independent ?

8. Suppose that X and Y are continuous random variables with joint pdf given by

$$f_{X,Y}(x,y) = cx(1-y) \qquad 0 < x < 1,\ 0 < y < 1$$

and zero otherwise for some constant c. Are X and Y independent random variables ?

Find the value of c., and, for the set $A \equiv \{(x,y) : 0 < x < y < 1\}$, the probability

$$P[X < Y] = \int_A\int f_{X,Y}(x,y)\ dxdy$$

9. Suppose that the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = 24xy \qquad x > 0,\ y > 0,\ x + y < 1$$

and zero otherwise. Find the marginal pdf of X, f_X.

Hint: Sketch the region on which the joint density is non-zero; remember that the integrand is only non-zero for some part of the integral range

10. Suppose that X and Y are continuous random variables with joint pdf given by

$$f_{X,Y}(x,y) = \frac{1}{2x^2y} \qquad 1 \leq x < \infty, 1/x \leq y \leq x$$

and zero otherwise. Derive the marginal pdf of X, the marginal pdf of Y, the conditional pdf of X given $Y = y$, and the conditional pdf of Y given $X = x$. Calculate the marginal expectation of Y, $E_{f_Y}[Y]$.