M2S1 - EXERCISES 3

CONTINUOUS PROBABILITY DISTRIBUTIONS

1. Show that the function, F_X, defined for $x \in \mathbb{R}$ by

$$F_X(x) = c \exp\left\{-e^{-\lambda x}\right\}$$

is a valid cdf for a continuous random variable X for a specific choice of constant c where parameter $\lambda > 0$. Find the pdf, f_X associated with this cdf.

Now consider the function $f_X(x) = cg(x)$ for some constant $c > 0$, with g defined by

$$g(x) = \frac{|x|}{(1+x^2)^2} \qquad x \in \mathbb{R}$$

Show that $f_X(x)$ is a valid pdf for a continuous random variable X with range $\mathbb{X} = \mathbb{R}$, and find the cdf, F_X, and the expected value of X, $E_{f_X}[X]$, associated with this pdf.

2. Let X be a continuous random variable with range $\mathbb{X} = \mathbb{R}^+$, pdf f_X and cdf F_X. By writing the expectation in its integral definition form on the left hand side, and changing the order of integration show that

$$E_{f_X}[X] = \int_0^\infty [1 - F_X(x)]\ dx$$

Using an identical approach, show also that for integer $r \geq 1$,

$$E_{f_X}[X^r] = \int_0^\infty r x^{r-1} [1 - F_X(x)]\ dx$$

Find a similar expression for random variables for which $\mathbb{X} = \mathbb{R}$.

3. (Harder) Suppose that continuous random variables X_1 and X_2 both with range $\mathbb{X} = \mathbb{R}^+$ have pdfs f_1 and f_2 respectively such that

$$\begin{aligned} f_1(x) &= c x^{-1} \exp\left\{-(\log(x))^2/2\right\} \qquad x > 0 \\ f_2(x) &= f_1(x)\left[1 + \sin(2\pi \log x)\right] \qquad x > 0 \end{aligned}$$

and $f_1(x) = f_2(x) = 0$ for $x \leq 0$. If, for $r = 1, 2, ...$, $E_{f_1}[X_1^r] = \exp\left\{r^2/2\right\}$, show that

$$E_{f_2}[X_2^r] = \exp\left\{r^2/2\right\}$$

Hint: write out the integral for $E_{f_2}[X_2^r]$*, and then make a transformation* $t = \log(x)$ *in the integral. Then complete the square.*

4. Suppose that X is a continuous random variable with range $\mathbb{R}$ and pdf given by

$$f_X(x) = \alpha^2 x \exp\{-\alpha x\} \qquad x \geq 0$$

and zero otherwise, for parameter $\alpha > 0$.

(i) Find the cdf of X, F_X, and hence show that, for any positive value m,

$$P[X \geq m] = (1 + \alpha m) \exp\{-\alpha m\}$$

(ii) Find $E_{f_X}[X]$. If the expected value of X is increased to $2/\beta$ (for $0 < \beta < \alpha$), find the associated change in $P[X \geq m]$.

5. Suppose that X is a continuous random variable with density function given by

$$f_X(x) = 4x^3 \qquad 0 < x < 1$$

and zero otherwise. Find the density functions of the following random variables

(a) $Y = X^4$ (b) $W = e^X$ (c) $Z = \log X$ (d) $U = (X - 0.5)^2$

Find the monotonic decreasing function H such that the random variable V, defined by $V = H(X)$, has a density function that is constant on the interval $(0, 1)$, and zero otherwise.

6. The measured radius of a circle, R, is a continuous random variable with density function given by

$$f_R(r) = 6r(1 - r) \qquad 0 < r < 1$$

and zero otherwise. Find the density functions of the circumference and the area of the circle.

7. (Harder) Suppose that X is a continuous random variable with density function given by

$$f_X(x) = \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} \qquad x > 0$$

for constants $\alpha, \beta > 0$, and zero otherwise. Find the density function and cdf of the random variable defined by $Y = \log X$, and the density function of the random variable defined by $Z = \xi + \theta Y$.