

## M2S1 : EXERCISE SHEET 4 : SOLUTIONS

1. Using this result for the given joint density, where random variables  $X, Y$  take values on the unit square, we have the marginal of  $U$  given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{X,Y}(v, u/v) |v|^{-1} dv = \int_u^1 c(\alpha, \beta) v^{\alpha-1} (1-v)^{\beta-1} c(\alpha + \beta, \gamma) \left(\frac{u}{v}\right)^{\alpha+\beta-1} \left(1 - \frac{u}{v}\right)^{\gamma-1} v^{-1} dv$$

as we have the density being positive only on the interval  $u < v < 1$  for fixed  $0 < u < 1$ , as  $XY < X$ . Then

$$\begin{aligned} f_U(u) &= \int_u^1 c(\alpha, \beta) v^{\alpha-1} (1-v)^{\beta-1} c(\alpha + \beta, \gamma) \left(\frac{u}{v}\right)^{\alpha+\beta-1} \left(1 - \frac{u}{v}\right)^{\gamma-1} \frac{1}{v} dv \\ &= c(\alpha, \beta) c(\alpha + \beta, \gamma) \int_u^1 \left(\frac{u}{t}\right)^{\alpha-1} \left(1 - \frac{u}{v}\right)^{\beta-1} t^{\alpha+\beta-1} (1-t)^{\gamma-1} \frac{t}{u} \frac{u}{t^2} dt \quad (t = u/v) \\ &= c(\alpha, \beta) c(\alpha + \beta, \gamma) \int_0^{1-u} \left(\frac{u}{s+u}\right)^{\alpha-1} \left(1 - \frac{u}{s+u}\right)^{\beta-1} (s+u)^{\alpha+\beta-2} (1-s-u)^{\gamma-1} ds \quad (s = t-u) \\ &= c(\alpha, \beta) c(\alpha + \beta, \gamma) \int_0^{1-u} u^{\alpha-1} s^{\beta-1} (1-s-u)^{\gamma-1} ds \\ &= c(\alpha, \beta) c(\alpha + \beta, \gamma) \int_0^1 u^{\alpha-1} r^{\beta-1} (1-u)^{\beta-1} (1-u)^{\gamma-1} (1-r)^{\gamma-1} (1-u) dr \quad (r = s/(1-u)) \\ &= c(\alpha, \beta) c(\alpha + \beta, \gamma) u^{\alpha-1} (1-u)^{\beta+\gamma-1} \int_0^1 r^{\beta-1} (1-r)^{\gamma-1} dr \\ &= c(\alpha, \beta) c(\alpha + \beta, \gamma) u^{\alpha-1} (1-u)^{\beta+\gamma-1} \frac{1}{c(\beta, \gamma)} = c(\alpha, \beta + \gamma) u^{\alpha-1} (1-u)^{\beta+\gamma-1} \quad 0 < u < 1 \end{aligned}$$

2. To compute the covariance need first the marginal expectations of  $X$  and  $Y$ . The key part of the solution is to realize that the support of the joint density is

$$0 < x < 1, 0 < y < 1, 0 < x + y < 1$$

that is, the “lower left corner” triangle of the unit square, bounded by the three lines  $x = 0, y = 0, x + y = 1$ . Now, for  $0 < x < 1$ ,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy = \int_0^{1-x} cxy(1-x-y) dy = cx \int_0^{1-x} y(1-x-y) dy \\ &= cx(1-x)^3 \int_0^1 t(1-t) dt \quad (t = y/(1-x)) \\ &= \frac{c}{6} x(1-x)^3 \quad 0 < x < 1 \end{aligned}$$

and

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_0^1 \frac{c}{6} x(1-x)^3 dx = 1 \implies c = 120$$

and hence

$$f_X(x) = 20x(1-x)^3 \quad 0 < x < 1 \quad \therefore E_{f_X}[X] = \int_0^1 20x^2(1-x)^3 dx = \frac{1}{3}$$

and, by symmetry,  $f_Y(y) = 20y(1-y)^3$  ( $0 < y < 1$ ),  $E_{f_Y}[Y] = \frac{1}{3}$ .

by symmetry. Also

$$\begin{aligned}
 \mathbf{E}_{f_{X,Y}}[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y) dx dy = \int_0^1 \left\{ \int_0^{1-y} 120x^2 y^2 (1-x-y) dx \right\} dy \\
 &= \int_0^1 120y^2 \left\{ \int_0^{1-y} x^2 (1-x-y) dx \right\} dy \\
 &= \int_0^1 120y^2 \left[ \frac{x^3}{3} (1-y) - \frac{x^4}{4} \right]_0^{1-y} dy \\
 &= \int_0^1 10y^2 (1-y)^4 dy \\
 &= 10 \left[ \frac{y^3}{3} - y^4 + \frac{6y^5}{5} - \frac{4y^6}{6} + \frac{y^7}{7} \right]_0^1 = 10 \left( \frac{1}{3} - 1 + \frac{6}{5} - \frac{2}{3} + \frac{1}{7} \right) = \frac{2}{21}
 \end{aligned}$$

and hence

$$\text{Cov}_{f_{X,Y}}[X, Y] = \mathbf{E}_{f_{X,Y}}[XY] - \mathbf{E}_{f_X}[X] \cdot \mathbf{E}_{f_Y}[Y] = \frac{2}{21} - \frac{1}{3} \cdot \frac{1}{3} = -\frac{1}{63}$$

3. Put  $U = X/Y$  and  $Z = X$ ; the inverse transformations are therefore  $X = Z$  and  $Y = Z/U$ , and note that the new variables are constrained by  $0 < Z < \min\{U, 1\}$ , as  $Y < 1$ . In terms of the multivariate transformation theorem, we have transformation functions defined by

$$\begin{aligned}
 g_1(t_1, t_2) &= t_1/t_2 & g_1^{-1}(t_1, t_2) &= t_2 \\
 g_2(t_1, t_2) &= t_1 & g_2^{-1}(t_1, t_2) &= t_2/t_1
 \end{aligned}$$

and the Jacobian of the transformation is given by

$$J(u, z) = \begin{vmatrix} 0 & 1 \\ -z/u^2 & 1/u \end{vmatrix} = \frac{z}{u^2}$$

and hence

$$f_{U,Z}(u, z) = f_{X,Y}(z, z/u) z/u^2 = z/u^2 \quad (u, z) \in \mathbb{U}^{(2)} = \{(u, z) : 0 < z < \min\{u, 1\}, u > 0\}$$

and zero otherwise, and so

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,Z}(u, z) dz = \int_0^{\min\{u, 1\}} z/u^2 dz = \frac{(\min\{u, 1\})^2}{2u^2} \quad u > 0.$$

Put  $V = -\log(XY)$  and  $Z = -\log X$ ; the inverse transformations are therefore  $X = e^{-Z}$  and  $Y = e^{-(v-z)}$ , and note that  $0 < Z < V$ . In terms of the theorem, we have transformation functions defined by

$$\begin{aligned}
 g_1(t_1, t_2) &= -\log(t_1 t_2) & g_1^{-1}(t_1, t_2) &= e^{-t_2} \\
 g_2(t_1, t_2) &= -\log t_1 & g_2^{-1}(t_1, t_2) &= e^{-(t_1 - t_2)}
 \end{aligned}$$

and the Jacobian of the transformation is given by

$$J(v, z) = \begin{vmatrix} 0 & -e^{-z} \\ -e^{-(v-z)} & e^{-(v-z)} \end{vmatrix} = e^{-v}$$

and hence

$$f_{V,Z}(v, z) = f_{X,Y}(e^{-z}, e^{-(v-z)}) e^{-v} = e^{-v} \quad (v, z) \in \mathbb{V}^{(2)} = \{(v, z) : 0 < z < v < \infty\}$$

and zero otherwise, and so

$$f_V(v) = \int_{-\infty}^{\infty} f_{V,Z}(v, z) dz = \int_0^v e^{-v} dz = ve^{-v} \quad v > 0$$

and zero otherwise. Note that we can attempt the joint transformation by setting

$$\begin{aligned} U &= X/Y \\ V &= -\log(XY) \end{aligned} \iff \begin{aligned} X &= U^{1/2}e^{-V/2} \\ Y &= U^{-1/2}e^{-V/2} \end{aligned}$$

note that, as  $X$  and  $Y$  lie in  $(0, 1)$  we have  $XY < X/Y$  and  $XY < Y/X$ , giving constraints  $e^{-V} < U$  and  $e^{-V} < 1/U$ , so that  $0 < e^{-V} < \min\{U, 1/U\}$ . The Jacobian of the transformation is

$$J(u, v) = \begin{vmatrix} \frac{u^{-1/2}e^{-v/2}}{2} & -\frac{u^{1/2}e^{-v/2}}{2} \\ -\frac{u^{-3/2}e^{-v/2}}{2} & -\frac{u^{-1/2}e^{-v/2}}{2} \end{vmatrix} = u^{-1}e^{-v}/2.$$

Hence

$$f_{U,V}(u, v) = u^{-1}e^{-v}/2 \quad 0 < e^{-v} < \min\{u, 1/u\}, \quad u > 0$$

The corresponding marginals are given below: let  $g(y) = -\log(\min\{u, 1/u\})$ , then

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v) dv = \int_{g(y)}^{\infty} \frac{e^{-v}}{2u} dv = \left[ -\frac{e^{-v}}{2u} \right]_{g(y)}^{\infty} = \frac{\min\{u, 1/u\}}{2u} \quad u > 0$$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u, v) du = \int_{e^{-v}}^{e^v} \frac{e^{-v}}{2u} du = \left[ \frac{\log u}{2} e^{-v} \right]_{e^{-v}}^{e^v} = ve^{-v} \quad v > 0$$

Now let

$$\begin{aligned} Y &= X_1 + X_2 & X_1 &= \frac{Y_1 + Y_2}{2} \\ Z &= X_1 - X_2 & X_2 &= \frac{Y_1 - Y_2}{2} \end{aligned} \iff$$

and the Jacobian of the transformation is  $1/2$ . The transformed variables take values on the square  $A$  with corners at  $(0, 0)$ ,  $(1, 1)$ ,  $(0, 2)$  and  $(1, -1)$  bounded by the lines  $z = 1 + y$ ,  $z = 1 - y$ ,  $z = -1 + y$  and  $z = -1 - y$ . Then

$$f_{Y,Z}(y, z) = \frac{1}{2} \quad (y, z) \in A$$

and zero otherwise (hint: sketch the square  $A$ ). Hence

$$f_Z(z) = \int_{-\infty}^{\infty} f_{Y,Z}(y, z) dy = \begin{cases} \int_{-z}^{2+z} \frac{1}{2} dy = 1+z & -1 < z \leq 0 \\ \int_z^{2-z} \frac{1}{2} dy = 1-z & 0 < z < 1 \end{cases}$$

4. The transformations are

$$\begin{aligned} Y_1 &= \frac{X_1}{X_1 + X_2 + X_3} & X_1 &= Y_1 Y_3 \\ Y_2 &= \frac{X_2}{X_1 + X_2 + X_3} & X_2 &= Y_2 Y_3 \\ Y_3 &= X_1 + X_2 + X_3 & X_3 &= Y_3(1 - Y_1 - Y_2) \end{aligned} \iff$$

which gives Jacobian

$$J(y_1, y_2, y_3) = \begin{vmatrix} y_3 & 0 & y_1 \\ 0 & y_3 & y_2 \\ -y_3 & -y_3 & (1 - y_1 - y_2) \end{vmatrix} = y_3^2$$

Hence the joint pdf is given by

$$\begin{aligned} f_{Y_1, Y_2, Y_3}(y_1, y_2, y_3) &= f_{X_1, X_2, X_3}(y_1 y_3, y_2 y_3, y_3(1 - y_1 - y_2)) J(y_1, y_2, y_3) \\ &= c_1 y_1 y_3 \exp\{-y_1 y_3\} c_2 y_2^2 y_3^2 \exp\{-y_2 y_3\} c_3 y_3^3 (1 - y_1 - y_2)^3 \exp\{-y_3(1 - y_1 - y_2)\} y_3^2 \\ &= c_1 c_2 c_3 y_1 y_2^2 (1 - y_1 - y_2)^3 y_3^8 \exp\{-y_3\} = f_{Y_1, Y_2}(y_1, y_2) f_{Y_3}(y_3) \end{aligned}$$

where

$$f_{Y_1, Y_2}(y_1, y_2) \propto y_1 y_2^2 (1 - y_1 - y_2)^3 \quad \text{and} \quad f_{Y_3}(y_3) \propto y_3^8 \exp\{-y_3\}$$

Hence  $Y_3 \sim \text{Gamma}(9, 1)$ ; the transformations give the constraints  $0 < Y_1, Y_2 < 1$  and  $0 < Y_1 + Y_2 < 1$ , and  $Y_3 > 0$ . Now

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1, Y_2}(y_1, y_2) dy_2 = \int_0^{1-y_1} c y_1 y_2^2 (1 - y_1 - y_2)^3 dy_2 = c y_1 (1 - y_1)^6 \int_0^1 t^2 (1 - t)^3 dt \quad (t = y_2 / (1 - y_1))$$

and hence

$$f_{Y_1}(y_1) \propto y_1 (1 - y_1)^6 \implies Y_1 \sim \text{Beta}(2, 7), \quad f_{Y_1}(y_1) = 336 y_1 (1 - y_1)^6 \quad 0 < y_1 < 1$$

and hence

$$\mathbb{E}_{f_{Y_1}}[Y_1] = \frac{2}{2+7} = \frac{2}{9}$$

as the expectation of a  $\text{Beta}(\alpha, \beta)$  distribution is  $\alpha / (\alpha + \beta)$  from notes.

5. Put  $U = X/Y$  and  $V = Y$ ; the inverse transformations are therefore  $X = UV$  and  $Y = V$ . In terms of the multivariate transformation theorem, we have transformation functions defined by

$$\begin{aligned} g_1(t_1, t_2) &= t_1 / t_2 & g_1^{-1}(t_1, t_2) &= t_1 t_2 \\ g_2(t_1, t_2) &= t_2 & g_2^{-1}(t_1, t_2) &= t_2 \end{aligned}$$

and the Jacobian of the transformation is given by

$$|J(u, v)| = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = |v|$$

and hence

$$f_{U, V}(u, v) = f_{X, Y}(uv, v) |v| = \left(\frac{1}{2\pi}\right) \exp\left\{-\frac{1}{2}(u^2 v^2 + v^2)\right\} |v| \quad (u, v) \in \mathbb{R}^2$$

and zero otherwise, and so, for any real  $u$ ,

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} f_{U, V}(u, v) dv = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right) \exp\left\{-\frac{1}{2}(u^2 v^2 + v^2)\right\} |v| dv \\ &= \left(\frac{1}{\pi}\right) \int_0^{\infty} v \exp\left\{-\frac{v^2}{2}(1 + u^2)\right\} dv && \text{integrand is even function} \\ &= \left(\frac{1}{\pi}\right) \left[ -\frac{1}{(1 + u^2)} \exp\left\{-\frac{v^2}{2}(1 + u^2)\right\} \right]_0^{\infty} && \text{by direct integration} \\ &= \frac{1}{\pi(1 + u^2)} \end{aligned}$$

Now put  $T = X/\sqrt{S/\nu}$  and  $R = S$ ; the inverse transformations are therefore  $X = T\sqrt{R/\nu}$  and  $S = R$ . In terms of the multivariate transformation theorem, we have transformation functions from  $(X, S) \rightarrow (T, R)$  defined by

$$\begin{aligned} g_1(t_1, t_2) &= t_1/\sqrt{t_2/\nu} & g_1^{-1}(t_1, t_2) &= t_1\sqrt{t_2/\nu} \\ g_2(t_1, t_2) &= t_2 & g_2^{-1}(t_1, t_2) &= t_2 \end{aligned}$$

and the Jacobian of the transformation is given by

$$|J(t, r)| = \begin{vmatrix} \sqrt{\frac{r}{\nu}} & \frac{t}{2\sqrt{r\nu}} \\ 0 & 1 \end{vmatrix} = \left| \sqrt{\frac{r}{\nu}} \right| = \sqrt{\frac{r}{\nu}}$$

and hence

$$f_{T,R}(t, r) = f_{X,S}\left(t\sqrt{\frac{r}{\nu}}, r\right) \sqrt{\frac{r}{\nu}} = f_X\left(t\sqrt{\frac{r}{\nu}}\right) f_S(r) \sqrt{\frac{r}{\nu}} \quad t \in \mathbb{R}, s \in \mathbb{R}^+$$

and zero otherwise, and so, for any real  $t$ ,

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{-\infty} f_{T,R}(t, r) dr \\ &= \int_0^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{rt^2}{2\nu}\right\} c(\nu)r^{\nu/2-1} e^{-r/2} \sqrt{\frac{r}{\nu}} dr \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \frac{c(\nu)}{\sqrt{\nu}} \int_0^{\infty} r^{(\nu+1)/2-1} \exp\left\{-\frac{r}{2}\left(1+\frac{t^2}{\nu}\right)\right\} dr \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \frac{c(\nu)}{\sqrt{\nu}} \left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \int_0^{\infty} z^{(\nu+1)/2-1} \exp\left\{-\frac{z}{2}\right\} dz \quad \text{setting } z = r\left(1+\frac{t^2}{\nu}\right) \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \frac{c(\nu)}{\sqrt{\nu}} \left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} \frac{1}{c(\nu+1)} \quad \text{integrand is a pdf} \end{aligned}$$

We also see/deduce that  $f_S$  is a *Gamma*( $\nu/2, 1/2$ ) or *Chiquared*( $\nu$ ) density, and that the normalizing constant  $c(\nu)$  is given by

$$c(\nu) = \frac{\left(\frac{1}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \implies f_T(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{1}{\pi\nu}\right)^{1/2} \frac{1}{(1+t^2/\nu)^{(\nu+1)/2}}$$

which is the *Student*( $\nu$ ) density.

6. We have

$$f_{X|Y}(x|y) = \sqrt{\frac{y}{2\pi}} \exp\left\{-\frac{yx^2}{2}\right\} \quad x \in \mathbb{R} \quad f_Y(y) = c(\nu)y^{\nu/2-1}e^{-\nu y/2} \quad y \in \mathbb{R}^+$$

where  $\nu$  is a positive integer, so that  $X|Y = y \sim N(0, y^{-1})$  and  $Y \sim \text{Gamma}(\nu/2, \nu/2)$ , and the normalizing constant  $c(\nu)$  is given by

$$c(\nu) = \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)}$$

Now, by the chain rule

$$f_{X,Y}(x, y) = f_{X|Y}(x|y)f_Y(y) \quad x \in \mathbb{R}, y \in \mathbb{R}^+$$

and zero otherwise, and so, for any real  $x$ ,

$$\begin{aligned}
 f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy \\
 &= \int_0^{\infty} \sqrt{\frac{y}{2\pi}} \exp\left\{-\frac{yx^2}{2}\right\} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} y^{\nu/2-1} e^{-\nu y/2} dy \\
 &= \frac{1}{\sqrt{2\pi}} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^{\infty} y^{(\nu+1)/2-1} \exp\left\{-\frac{y}{2}(\nu+x^2)\right\} dy \\
 &= \frac{1}{\sqrt{2\pi}} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\left(\frac{1}{2}(\nu+x^2)\right)^{(\nu+1)/2}} \quad \text{integrand } \propto \text{ a Gamma pdf}
 \end{aligned}$$

Therefore  $f_X$  is given by

$$f_X(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{1}{\pi\nu}\right)^{1/2} \frac{1}{(1+x^2/\nu)^{(\nu+1)/2}}$$

which is again the *Student*( $\nu$ ) density.

Exercises 5 and 6 give the two alternative ways of specifying the Student-t distribution, either as a function of independent Normal and Gamma/Chi-squared variables, or as the marginal obtained by “*scale-mixing*” a Normal distribution by a Gamma distribution (that is, rather than having a fixed variance  $\sigma^2 = 1/Y$ , we regard  $Y$  as a *random variable* having a Gamma distribution, so that  $(X, Y)$  have a joint distribution

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y)$$

from which we calculate  $f_X(x)$  by integration.

7. Can calculate the mgf for this mass function (the Poisson distribution) as

$$M_X(t) = \exp\{\lambda(e^t - 1)\}.$$

Now, if  $Z_1 = (X - \lambda)/\sqrt{\lambda}$ , we use the mgf result for linear functions, that is if  $Y = aX + b$ ,  $M_X(t) = e^{bt}M_X(at)$ . Here,  $a = 1/\sqrt{\lambda}$  and  $b = -\sqrt{\lambda}$ , so

$$\begin{aligned}
 M_{Z_1}(t) &= e^{-\sqrt{\lambda}t} \exp\left\{\lambda(e^{t/\sqrt{\lambda}} - 1)\right\} = \exp\left\{-\sqrt{\lambda}t + \lambda\left[\frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{6\sqrt{\lambda}} + \dots\right]\right\} \\
 &= \exp\left\{\frac{t^2}{2} + \frac{t^3}{6\sqrt{\lambda}} + \dots\right\} \rightarrow \exp\left\{\frac{t^2}{2}\right\} \quad \text{as } \lambda \rightarrow \infty
 \end{aligned}$$

so therefore

$$Z_1 \rightarrow Z \sim \text{Normal}(0, 1) \quad \text{as } \lambda \rightarrow \infty$$

that is, the distribution  $Z_1$  tends to a standard Normal distribution as  $\lambda \rightarrow \infty$ . Re-arranging the definition of  $Z_1$ , and using the transformation result for Normal random variables

$$X \sim \text{Normal}(0, 1), Y = aX + b \implies Y \sim \text{Normal}(b, a^2)$$

we have that

$$X = \sqrt{\lambda}Z_1 + \lambda \sim \text{Normal}(\lambda, \lambda)$$

Thus we have that

$$\text{Poisson}(\lambda) \approx \text{Normal}(\lambda, \lambda) \quad \text{if } \lambda \text{ is large}$$

For second part, can calculate the mgf for this pdf function (the Gamma distribution with parameters  $\lambda$  and 1) as

$$M_X(t) = \left( \frac{1}{1-t} \right)^\lambda.$$

Now, if  $Z_2 = (X - \lambda)/\sqrt{\lambda}$ , we use the mgf result for linear functions, that is if  $Y = aX + b$ ,  $M_X(t) = e^{bt} M_X(at)$ . Here,  $a = 1/\sqrt{\lambda}$  and  $b = -\sqrt{\lambda}$ , so

$$\begin{aligned} M_{Z_2}(t) &= e^{-\sqrt{\lambda}t} \left( \frac{1}{1-t/\sqrt{\lambda}} \right)^\lambda = \exp \left\{ -\sqrt{\lambda}t - \lambda \log \left( 1 - \frac{t}{\sqrt{\lambda}} \right) \right\} \\ &= \exp \left\{ -\sqrt{\lambda}t + \lambda \left[ \frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{3^{3/2}\sqrt{\lambda}} + \dots \right] \right\} \\ &= \exp \left\{ \frac{t^2}{2} + \frac{t^3}{3\sqrt{\lambda}} + \dots \right\} \rightarrow \exp \left\{ \frac{t^2}{2} \right\} \quad \text{as } \lambda \rightarrow \infty \end{aligned}$$

so therefore

$$Z_2 \rightarrow Z \sim \text{Normal}(0, 1) \quad \text{as } \lambda \rightarrow \infty$$

that is, the distribution  $Z_2$  tends to a standard Normal distribution as  $\lambda \rightarrow \infty$ . Re-arranging the definition of  $Z_2$ , and using the transformation result for Normal random variables as above, we have that

$$X = \sqrt{\lambda}Z_2 + \lambda \sim \text{Normal}(\lambda, \lambda)$$

Thus we have that

$$\text{Gamma}(\lambda, 1) \approx \text{Normal}(\lambda, \lambda) \quad \text{if } \lambda \text{ is large}$$

8. If  $Y_2 = \max\{X_1, X_2\}$  then  $P[Y_2 > c] = 1 - p^2$ , as

$$P[Y_2 \leq c] = P[\max\{X_1, X_2\} \leq c] = P[(X_1 \leq c) \cap (X_2 \leq c)] = P[X_1 \leq c]P[X_2 \leq c] = p^2$$

9. If  $Y_1 = \min\{X_1, \dots, X_k\}$ , then

$$\begin{aligned} F_{Y_1}(y_1) &= P[Y_1 \leq y_1] = P[\min\{X_1, \dots, X_k\} \leq y_1] \\ &= 1 - P[\min\{X_1, \dots, X_k\} > y_1] = 1 - P[(X_1 > y_1) \cap \dots \cap (X_k > y_1)] \\ &= 1 - P[(X_1 > y_1)] \dots P[(X_k > y_1)] = 1 - (1 - F_X(y_1)) \dots (1 - F_X(y_1)) = 1 - \{(1 - F_X(y_1))\}^k \end{aligned}$$

Now here,  $f_X(x) = \lambda e^{-\lambda x}$ , so  $F_X(x) = 1 - e^{-\lambda x}$ , when  $x > 0$ , and hence

$$F_{Y_1}(y_1) = 1 - \{(1 - F_X(y_1))\}^k = 1 - \{1 - (1 - e^{-\lambda y_1})\}^k = 1 - e^{-\lambda y_1} \implies f_{Y_1}(y_1) = k\lambda e^{-k\lambda y_1} \quad y_1 > 0.$$

.In part two, the cdf for  $X$  is

$$F_X(x) = \int_{-\infty}^x f_X(t) dt = \int_1^x \frac{1}{t^2} dt = 1 - \frac{1}{x} = \frac{x-1}{x} \quad 1 \leq x \leq \infty$$

Using Result 1 of the Order Statistics section: the joint distribution of the order statistics  $Y_1, \dots, Y_k$  derived from  $X_1, \dots, X_k$  is

$$f_{Y_1, \dots, Y_k}(y_1, \dots, y_k) = k! f_{X_1, \dots, X_k}(x_1, \dots, x_k) = k! \prod_{i=1}^k \frac{1}{y_i^2}$$

and by Result 2, or by direct calculation as above, the marginal pdfs of minimum  $Y_1$  and maximum  $Y_k$  are given by

$$f_{Y_1}(y_1) = k \left\{ \frac{1}{y_1} \right\}^{k-1} \frac{1}{y_1^2} = \frac{k}{y_1^{k+1}} \quad f_{Y_k}(y_k) = k \left\{ \frac{y_k - 1}{y_k} \right\}^{k-1} \frac{1}{y_k^2} = \frac{k(y_k - 1)^{k-1}}{y_k^{k+1}}$$

with support  $[1, \infty)$  in each case.

10. We have  $F_X(x) = x$ , and  $f_X(x) = 1$  for  $0 < x < 1$ . Either by direct calculation, or by using the Order Statistics Result 2, we have

$$F_{Y_k}(y_k) = \{F_X(y_k)\}^k = y_k^k \quad f_{Y_k}(y_k) = \frac{k!}{0!(k-1)!} \{F_X(y_k)\}^{k-1} \{1 - F_X(y_k)\}^0 f_X(y_k) = ky_k^{k-1}$$

so that

$$P[Y_k \geq 0.99] = 1 - P[Y_k < 0.99] = 1 - F_{Y_k}(0.99) = 1 - (0.99)^k$$

and the smallest  $k$  such that  $1 - (0.99)^k \geq 0.95$  is  $k = 295$ .

11. For this pdf, we have cdf  $F_X(x) = x^5$ ,  $0 < x < 1$ . From elementary calculations, we have that

$$F_{Y_1}(y_1) = 1 - \{1 - F_X(y_1)\}^k = 1 - (1 - y_1^5)^k \quad 0 < y_1 < 1.$$

Therefore,  $P[Y_1 \leq 0.75] = 1 - (1 - (0.75)^5)^k$ , which takes the values 0.556, 0.662, 0.742 for  $k = 3, 4, 5$ . Also

$$f_{Y_1}(y_1) = k \{1 - F_X(y_1)\}^{k-1} f_X(y_1) = 5ky_1^4 \{1 - y_1^5\}^{k-1} \quad 0 \leq y_1 \leq 1$$

and

$$F_{Y_1}(y_1) = 1 - \{1 - y_1^5\}^k = 1 - (1 - y_1^5)^k \quad \longrightarrow \quad \begin{cases} 0 & y_1 = 0 \\ 1 & y_1 > 1 \end{cases}$$

as  $k \rightarrow \infty$ , so that the limiting distribution is a step function, with the step at zero. That is, the limiting distribution has mass one at  $Y_1 = 0$ .

12. We have that

$$f_X(x) = \frac{(\alpha + 1)x^\alpha}{\theta^{\alpha+1}} \quad F_X(x) = \left(\frac{x}{\theta}\right)^{\alpha+1} \quad 0 \leq x \leq \theta$$

and using the results for maxima and minima, we have that

$$\begin{aligned} F_{Y_1}(y) &= 1 - \{1 - F_X(y)\}^n = 1 - \left\{1 - \left(\frac{y}{\theta}\right)^{\alpha+1}\right\}^n \quad 0 \leq x \leq \theta \\ &= \begin{cases} 0 & y < 0 \\ 1 & y \geq 0 \end{cases} \quad \text{as } n \rightarrow \infty \text{ with } y \text{ fixed.} \end{aligned}$$

(as the second term is 1 if  $y = 0$ , but is zero otherwise) that is, the limiting distribution (cdf) as  $n \rightarrow \infty$  is a step function, with a single step at  $y = 0$ . Hence the limiting distribution is *degenerate* at  $y = 0$ , that is, we have  $Y_1$  converging in distribution to a discrete random variable  $Y$  with  $P[Y = 0] = 1$ .

Similarly, for the maximum order statistic,  $Y_n$  we have

$$\begin{aligned} F_{Y_n}(y) &= \{F_X(y)\}^n = \left\{\left(\frac{y}{\theta}\right)^{\alpha+1}\right\}^n = \left(\frac{y}{\theta}\right)^{n(\alpha+1)} \quad 0 \leq x \leq \theta \\ &= \begin{cases} 0 & y < \theta \\ 1 & y \geq \theta \end{cases} \quad \text{as } n \rightarrow \infty \text{ with } y \text{ fixed} \end{aligned}$$

(as  $y/\theta$  is less than 1 if  $y < \theta$ ) that is, the limiting distribution (cdf) as  $n \rightarrow \infty$  is a step function, with a single step at  $y = \theta$ . Hence the limiting distribution is *degenerate* at  $y = \theta$ , that is, we have  $Y_n$  converging in distribution to a discrete random variable  $Y$  with  $P[Y = \theta] = 1$ .