M2S1 : ASSESSED COURSEWORK 3 : SOLUTIONS

1. (a) We have $U \sim Uniform(0,1)$ and

$$X_1 = \begin{cases} \dfrac{1}{\lambda}\log 2U & 0 < U \leq \dfrac{1}{2} \\ \\ -\dfrac{1}{\lambda}\log(2-2U) & \dfrac{1}{2} < U < 1 \end{cases}$$

for $\lambda > 0$. For $x \leq 0$ $\left(\text{so that } 0 < u \leq \frac{1}{2}\right)$

$$F_{X_1}(x) = P\left[X_1 \leq x\right] = P\left[\frac{1}{\lambda}\log 2U \leq x\right] = P\left[U \leq \frac{1}{2}e^{\lambda x}\right] = \frac{1}{2}e^{\lambda x}$$

and for $x > 0$ $\left(\text{so that } \frac{1}{2} < u < 1\right)$

$$\begin{aligned} F_{X_1}(x) &= P\left[X_1 \leq x\right] = P\left[X_1 \leq 0\right] + P\left[0 < X_1 \leq x\right] \\ &= \frac{1}{2} + P\left[0 < -\frac{1}{\lambda}\log(2-2U) \leq x\right] \\ &= \frac{1}{2} + P\left[\frac{1}{2}e^{-\lambda x} \leq 1 - U < \frac{1}{2}\right] \\ &= \frac{1}{2} + P\left[\frac{1}{2} < U \leq 1 - \frac{1}{2}e^{-\lambda x}\right] \\ &= \frac{1}{2} + 1 - \frac{1}{2}e^{-\lambda x} - \frac{1}{2} = 1 - \frac{1}{2}e^{-\lambda x} \end{aligned}$$

Hence, by differentiation

$$f_{X_1}(x) = \begin{cases} \dfrac{\lambda}{2}e^{\lambda x} & 0 \leq x \\ \\ \dfrac{\lambda}{2}e^{-\lambda x} & x > 0 \end{cases} = \frac{\lambda}{2}e^{-\lambda|x|} \qquad x \in \mathbb{R}$$

[4 MARKS]

(ii) For the mgf

$$\begin{aligned} E_{f_{X_1}}\left[e^{tX_1}\right] &= \int_{-\infty}^{\infty} e^{tx} f_{X_1}(x)dx = \int_{-\infty}^{\infty} e^{tx}\frac{\lambda}{2}e^{-\lambda|x|}dx \\ &= \frac{\lambda}{2}\left[\int_{-\infty}^{0} e^{(\lambda+t)x}dx + \int_{0}^{\infty} e^{-(\lambda-t)x}dx+\right] \\ &= \frac{\lambda}{2}\left[\frac{1}{\lambda+t} + \frac{1}{\lambda-t}\right] \qquad \text{if } |t| < \lambda \\ &= \frac{\lambda^2}{(\lambda+t)(\lambda-t)} = \frac{\lambda^2}{\left(\lambda^2-t^2\right)} = \frac{1}{1-\dfrac{t^2}{\lambda^2}} \qquad (1) \end{aligned}$$

[6 MARKS]

(b) The joint pdf has support $\mathbb{X}^{(2)} = \mathbb{R} \times \mathbb{R}^{+}$

$$f_{X_2,Y}(x,y) = f_{X_2|Y}(x|y) f_Y(y) = \left(\frac{1}{2\pi y}\right)^{1/2} \exp\left\{-\frac{x^2}{2y}\right\} \gamma \exp\left\{-\gamma y\right\}$$

(i) To compute the mgf, use the Law of Iterated Expectation

$$E_{f_{X_2}}\left[e^{tX_2}\right] = E_{f_Y}\left[E_{f_{X_2|Y}}\left[e^{tX_2}|Y=y\right]\right].$$

Now

$$E_{f_{X_2|Y}}\left[e^{tX_2}|Y=y\right] = \exp\left\{\frac{yt^2}{2}\right\}$$

as $X_2|Y = y \sim Normal\,(0, y)$, using the mgf of a Normal random variable from the formula sheet, with $\mu = 0$ and $\sigma^2 = y$. Thus

$$E_{f_{X_2}}\left[e^{tX_2}\right] = E_{f_Y}\left[\exp\left\{\frac{Yt^2}{2}\right\}\right] = M_Y\left(\frac{t^2}{2}\right) = \frac{\gamma}{\gamma - \dfrac{t^2}{2}}$$

as $Y \sim Exponential\,(\gamma)$, using the mgf for am Exponential from the formula sheet. Hence

$$M_{X_2}(t) = \frac{\gamma}{\gamma - \dfrac{t^2}{2}} = \frac{1}{1 - \dfrac{t^2}{2\gamma}} \tag{2}$$

[4 MARKS]

(ii) Comparing (1) and (2), and setting $\lambda^2 = 2\gamma$, we can deduce that

$$f_{X_2}(x) = \frac{\sqrt{2\gamma}}{2} \exp\left\{-\sqrt{2\gamma}\,|x|\right\} \qquad x \in \mathbb{R}$$

by the uniqueness of mgfs.

[4 MARKS]

(iii) Using properties of mgfs

$$E_{f_{X_2}}\left[X_2^r\right] = \frac{d^r}{dt^r}\left\{M_{X_2}(t)\right\}_{t=0}$$

We have

$$\frac{d}{dt}\left\{M_{X_2}(t)\right\}_{t=0} = \left\{\frac{4\gamma t}{(2\gamma - t^2)^2}\right\}_{t=0} = 0 \qquad \therefore \qquad E_{f_{X_2}}\left[X_2\right] = 0$$

$$\frac{d^2}{dt^2}\left\{M_{X_2}(t)\right\}_{t=0} = \left\{\frac{4\gamma\left(2\gamma - t^2\right)^2 + 16\gamma t^2}{\left(2\gamma - t^2\right)^4}\right\}_{t=0} = \frac{16\gamma^3}{16\gamma^4} = \frac{1}{\gamma} \qquad \therefore \qquad E_{f_{X_2}}\left[X_2^2\right] = \frac{1}{\gamma}$$

so therefore

$$Var_{f_{X_2}}\left[X_2\right] = E_{f_{X_2}}\left[X_2^2\right] - \left\{E_{f_{X_2}}\left[X_2\right]\right\}^2 = \frac{1}{\gamma}$$

[2 MARKS]

2. (a)(i) $X_1, X_2, ...X_n \sim Poisson(\lambda)$. From the formula sheet we have the mgf of each X_i equal to

$$\exp\left\{\lambda\left(e^t - 1\right)\right\}$$

and hence, if $T_n = \sum_{i=1}^{n} X_i$, by the key mgf result

$$M_{T_n}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \exp\left\{\lambda\left(e^t - 1\right)\right\} = \exp\left\{n\lambda\left(e^t - 1\right)\right\}$$

and thus, by uniqueness of mgfs

$$T_n \sim Poisson(n\lambda)$$

[2 MARKS]

(ii) Standardized variable

$$Z_n = \frac{T_n - n\lambda}{\sqrt{n\lambda}}$$

and by the Central Limit Theorem, we have

$$Z_n \xrightarrow{d} Z \sim N(0,1)$$

and hence, by location/scale transformation of Z_n, for large n

$$T_n \dot{\sim} N(n\lambda, n\lambda)$$

so that

$$F_{T_n}(t) = P\left[T_n \le t\right] = P\left[\frac{T_n - n\lambda}{\sqrt{n\lambda}} \le \frac{t - n\lambda}{\sqrt{n\lambda}}\right] \approx \Phi\left(\frac{t - n\lambda}{\sqrt{n\lambda}}\right)$$

[4 MARKS]

(b)(i) We have, from the definitions given, that for $i = 0, 1$

$$X_i \sim Binomial\left(n_i, \theta_i\right)$$

and consequently

$$X_i = \sum_{j=1}^{n_i} X_{ij}$$

where $X_{ij} \sim Bernoulli\left(\theta_i\right)$, independently, for $j = 1, 2, ..., n_i$ so that, from the formula sheet

$$E_{f_{X_i}}\left[X_i\right] = \theta_i \qquad Var_{f_{X_i}}\left[X_i\right] = \theta_i\left(1 - \theta_i\right).$$

Now, $i = 0, 1$,

$$Z_i = \frac{X_i - n_i\theta_i}{\sqrt{n_i\theta_i\left(1 - \theta_i\right)}}$$

is a standardized variable, and hence by the Central Limit Theorem we have

$$Z_i \xrightarrow{d} Z \sim N(0,1)$$

as $n_i \to \infty$.

[4 MARKS]

(ii) If $Z \sim N(0,1)$, then

$$M_Z(t) = \exp\left\{\frac{t^2}{2}\right\}$$

so that the mgf of $Z_1 - Z_0$ (in the limit as $n_0 \to \infty$ and $n_1 \to \infty$) is

$$M_{Z_1}(t) \times M_{Z_0}(-t) \exp\left\{\frac{t^2}{2}\right\} \times \exp\left\{\frac{(-t)^2}{2}\right\} = \exp\left\{\frac{2t^2}{2}\right\} = \exp\left\{\frac{\left(\sqrt{2}t\right)^2}{2}\right\}$$

by a key mgf result, and the result for linear transformations of Normal random variables. Hence

$$Z_1 - Z_0 \dot{\sim} N\left(0, \left(\sqrt{2}\right)^2\right) \equiv N(0,2)$$

[2 MARKS]

(iii) Can quote result from lecture notes, but from first principles, if $Z \sim N(0,1)$ and $V = Z^2$ then for $v > 0$

$$F_V(v) = P[V \leq v] = P\left[X^2 \leq v\right] = P\left[-\sqrt{v} \leq X \leq \sqrt{v}\right] = F_X(\sqrt{v}) - F_X(-\sqrt{v}) = \Phi(\sqrt{v}) - \Phi(-\sqrt{v})$$

and so, by differentiation

$$f_V(v) = \frac{1}{2\sqrt{v}}\left[\phi\left(\sqrt{v}\right) + \phi\left(-\sqrt{v}\right)\right] = \frac{1}{\sqrt{v}}\left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(\sqrt{v}\right)^2\right\} = \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma\left(\frac{1}{2}\right)} v^{1/2-1} \exp\left\{-\frac{v}{2}\right\} \qquad v > 0$$

so from the formula sheet

$$V \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right) \equiv \chi_1^2 \qquad \therefore \qquad M_V(t) = \frac{\frac{1}{2}}{\frac{1}{2} - t}$$

Hence, for large n_0 and n_1 $Q_0 \dot{\sim} \chi_1^2$ and $Q_1 \dot{\sim} \chi_1^2$ and thus using mgfs

$$M_Q(t) = M_{Q_0}(t) M_{Q_1}(t) \approx \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right) \times \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^2$$

so that, from the formula sheet

$$Q \dot{\sim} Gamma\left(\frac{2}{2}, \frac{1}{2}\right) \equiv Gamma\left(1, \frac{1}{2}\right) \equiv \chi_2^2 \equiv Exponential\left(\frac{1}{2}\right)$$

[4 MARKS]

(iv) If $\theta_0 = \theta_1 = \theta$ say, then from the formula sheet

$$M_Y(y) = M_{X_0}(t) M_{X_1}(t) = \left(1 - \theta + \theta e^t\right)^{n_1} \times \left(1 - \theta + \theta e^t\right)^{n_2} = \left(1 - \theta + \theta e^t\right)^{n_1 + n_2}$$

and hence

$$Y \sim Binomial\left(n_0 + n_1, \theta\right)$$

so that, by the Central Limit Theorem

$$Y \dot{\sim} Normal\left(\left(n_0 + n_1\right)\theta, \left(n_0 + n_1\right)\theta\left(1 - \theta\right)\right)$$

[4 MARKS]