MULTIVARIATE DISTRIBUTIONS - WORKED EXAMPLES

EXAMPLE 1 Let X_1 and X_2 be discrete random variables each with range $\{1, 2, 3, ...\}$ and joint mass function

$$f_{X_1,X_2}(x_1, x_2) = \frac{c}{(x_1 + x_2 - 1)(x_1 + x_2)(x_1 + x_2 + 1)} \qquad x_1, x_2 = 1, 2, 3, ...$$

and zero otherwise. The marginal mass function for X is given by

$$\begin{aligned} f_{X_1}(x_1) &= \sum_{x_2=-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2) = \sum_{x_2=1}^{\infty} \frac{c}{(x_1 + x_2 - 1)(x_1 + x_2)(x_1 + x_2 + 1)} \\ &= \sum_{x_2=1}^{\infty} \frac{c}{2} \left[\frac{1}{(x_1 + x_2 - 1)(x_1 + x_2)} - \frac{1}{(x_1 + x_2)(x_1 + x_2 + 1)} \right] = \frac{c}{2} \frac{1}{x_1(x_1 + 1)} \end{aligned}$$

as all other terms cancel, and to calculate c, note that

$$\sum_{x_1=-\infty}^{\infty} f_{X_1}(x_1) = \sum_{x_1=1}^{\infty} \frac{c}{2} \frac{1}{x_1(x_1 + 1)} = \frac{c}{2} \sum_{x_1=1}^{\infty} \left[\frac{1}{x_1} - \frac{1}{x_1 + 1} \right] = \frac{c}{2}$$

as all terms in the sum except the first cancel. Hence $c = 2$, and by symmetry f_{X_1} and f_{X_2} are identical.

EXAMPLE 2 Let X_1 and X_2 be continuous random variables with ranges $\mathbb{X}_1 = \mathbb{X}_2 = (0, 1)$ and joint pdf defined by

$$f_{X_1,X_2}(x_1, x_2) = 4x_1x_2 \qquad 0 < x_1 < 1,\ 0 < x_2 < 1$$

and zero otherwise. For $0 < x_1, x_2 < 1$,

$$\begin{aligned} F_{X_1,X_2}(x_1, x_2) &= \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_{X_1,X_2}(t_1, t_2)\ dt_1 dt_2 = \int_0^{x_2} \int_0^{x_1} 4t_1t_2\ dt_1 dt_2 \\ &= \left\{ \int_0^{x_1} 2t_1\ dt_1 \right\} \left\{ \int_0^{x_2} 2t_2\ dt_2 \right\} = (x_1x_2)^2 \end{aligned}$$

and a full specification for F_{X_1,X_2} is

$$F_{X_1,X_2}(x_1, x_2) = \begin{cases} 0 & x_1, x_2 \le 0 \\ (x_1x_2)^2 & 0 < x_1, x_2 < 1 \\ x_1^2 & 0 < x_1 < 1, x_2 \ge 1 \\ x_2^2 & 0 < x_2 < 1, x_1 \ge 1 \\ 1 & x_1, x_2 \ge 1 \end{cases}$$

To calculate P[$(X_1 + X_2)/2 < c$], need to integrate f_{X_1,X_2} over the set

$$A = \{\ (x_1, x_2)\ :\ 0 < x_1, x_2 < 1, (x_1 + x_2)/2 < c\ \},$$

that is, if $c = 1/2$,

$$\mathrm{P}[\ (X_1 + X_2) < 1\ ] = \int_0^1 \int_0^{1-x_1} 4x_1x_2\ dx_1 dx_2 = \int_0^1 2x_1(1 - x_1)^2\ dx_1 = \frac{1}{6}$$

[Hint: before trying to calculate the probability, sketch the joint range and the set A].

EXAMPLE 3 Let X_1, X_2 be continuous random variables with ranges $\mathbb{X}_1 = \mathbb{X}_2 = [0,1]$, and joint pdf defined by

$$f_{X_1,X_2}(x_1,x_2) = 1 \qquad 0 \le x_1, x_2 \le 1$$

and zero otherwise. Let $Y = X_1 + X_2$. Then

$$F_Y(y) = \mathrm{P}[\ Y \le y\ ] = \mathrm{P}[\ X_1 + X_2 \le y\ ]$$

Clearly Y has range $\mathbb{Y} = [0,2]$, Now, to calculate $\mathrm{P}[\ (X_1 + X_2) \le y\ ]$, need to integrate f_{X_1,X_2} over the set $A_y = \{\ (x_1,x_2)\ :\ 0 < x_1, x_2 < 1, x_1 + x_2 \le y\ \}$. Now for $0 \le y \le 1$,

$$\mathrm{P}[\ (X_1 + X_2) < y\ ] = \int_0^y \int_0^{y-x_2} 1\ dx_1 dx_2 = \int_0^y (y - x_2)\ dx_2 = \frac{y^2}{2}$$

whereas for $1 \le y \le 2$, we have

$$\mathrm{P}[\ X_1 + X_2 \le y\ ] = 1 - \mathrm{P}[\ X_1 + X_2 > y\ ] = 1 - \int_{y-1}^1 \int_{y-x_2}^1 1\ dx_1 dx_2 = 1 - \int_{y-1}^1 (1 - y + x_2)\ dx_2 = -\frac{y^2}{2} + 2y - 1$$

These two expressions give the cdf F_Y, and hence by differentiation we have

$$f_Y(y) = \begin{cases} y & 0 \le y \le 1 \\ 2 - y & 1 \le y \le 2 \end{cases}$$

and zero otherwise.

[Hint: before trying to calculate the probabilities, sketch the set A_y in the two cases $0 \le y \le 1$ and $1 \le y \le 2$].

EXAMPLE 4 Let X_1 and X_2 be continuous random variables with ranges $\mathbb{X}_1 = (0,1)$, $\mathbb{X}_2 = (0,2)$ and joint pdf defined by

$$f_{X_1,X_2}(x_1,x_2) = c\left(x_1^2 + \frac{x_1 x_2}{2}\right) \qquad 0 < x_1 < 1,\ 0 < x_2 < 2$$

and zero otherwise. To calculate c, we have

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2) dx_1 dx_2 &= \int_0^2 \int_0^1 c\left(x_1^2 + \frac{x_1 x_2}{2}\right) dx_1 dx_2 = \int_0^2 c\left[\frac{x_1^3}{3} + \frac{x_1^2 x_2}{4}\right]_0^1 dx_2 \\ &= \int_0^2 c\left(\frac{1}{3} + \frac{x_2}{4}\right) dx_2 = c\left[\frac{x_2}{3} + \frac{x_2^2}{8}\right]_0^2 = c\frac{7}{6} \end{aligned}$$

so $c = 6/7$. The marginal pdf of X_1 is given by

$$\begin{aligned} f_{X_1}(x_1) &= \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\ dx_2 = \int_0^2 \frac{6}{7}\left(x_1^2 + \frac{x_1 x_2}{2}\right)\ dx_2 \\ &= \frac{6}{7}\left[x_1^2 x_2 + \frac{x_1 x_2^2}{4}\right]_0^2 = \frac{6x_1(2x_1 + 1)}{7} \qquad 0 < x_1 < 1 \end{aligned}$$

and zero otherwise. To compute $\mathrm{P}[\ X_1 > X_2\ ]$, let $A = \{\ (x_1,x_2)\ :\ 0 < x_1 < 1, 0 < x_2 < 2, x_2 < x_1\ \}$, so

$$\begin{aligned} \mathrm{P}[\ X_1 > X_2\ ] &= \iint_A f_{X_1,X_2}(x_1,x_2)\ dx_2 dx_1 = \int_0^1 \int_0^{x_1} \frac{6}{7}\left(x_1^2 + \frac{x_1 x_2}{2}\right)\ dx_2 dx_1 \\ &= \int_0^1 \left[x_1^2 x_2 + \frac{x_1 x_2^2}{4}\right]_0^{x_1}\ dx_1 = \int_0^1 \left(x_1^3 + \frac{x_1^3}{4}\right)\ dx_1 = \frac{6}{7}\left[\frac{5x_1^4}{16}\right]_0^1 = \frac{15}{56} \end{aligned}$$

EXAMPLE 5 Let X_1, X_2 and X_3 be continuous random variables with joint ranges

$$\mathbb{X}^{(3)} = \{ \ (x_1, x_2, x_3) \ : \ 0 < x_1 < x_2 < x_3 < 1\}$$

and joint pdf defined by

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = c \qquad 0 < x_1 < x_2 < x_3 < 1$$

and zero otherwise. To calculate c, integrate carefully over $\mathbb{X}^{(3)}$, that is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(x_1, x_2, x_3) \ dx_1 \ dx_2 \ dx_3 = 1 \Longrightarrow \int_0^1 \left\{ \int_0^{x_3} \left\{ \int_0^{x_2} c \ dx_1 \right\} \ dx_2 \right\} \ dx_3 = 1.$$

Now

$$\int_0^1 \left\{ \int_0^{x_3} \left\{ \int_0^{x_2} c \ dx_1 \right\} \ dx_2 \right\} \ dx_3 = \int_0^1 \left\{ \int_0^{x_3} c x_2 \ dx_2 \right\} \ dx_3 = \int_0^1 \frac{c x_3^2}{2} \ dx_3 = \frac{c}{6}$$

and hence $c = 6$.

Also, for $0 < x_3 < 1$,

$$f_{X_3}(x_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(x_1, x_2, x_3) \ dx_1 \ dx_2 = \int_0^{x_3} \left\{ \int_0^{x_2} 6 \ dx_1 \right\} \ dx_2 = \int_0^{x_3} 6x_2 \ dx_2 = 3x_3^2$$

and zero otherwise. Similar calculations give

$$\begin{array}{ll} f_{X_1}(x_1) = 3(1 - x_1)^2 & 0 < x_1 < 1 \\ f_{X_2}(x_2) = 6x_2(1 - x_2) & 0 < x_2 < 1 \end{array}$$

and zero otherwise, and furthermore

$$f_{X_1,X_2}(x_1, x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(x_1, x_2, x_3) \ dx_3 = \int_{x_2}^1 6 \ dx_3 = 6(1 - x_2) \qquad 0 < x_1 < x_2 < 1$$

and zero otherwise. Combining these results, we have, for example

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_2}(x_2)} = \frac{1}{x_2} \qquad 0 < x_1 < x_2$$

and zero otherwise for **fixed** x_2.
We can calculate the expectation of X_1 either directly or using the Law of Iterated Expectation: we have

$$\mathrm{E}_{f_{X_1}} [\ X_1 \ ] = \int_{-\infty}^{\infty} x_1 f_{X_1}(x_1) \ dx_1 = \int_0^1 x_1 \ 3(1 - x_1)^2 \ dx_1 = \frac{1}{4}$$

or, alternatively,

$$\mathrm{E}_{f_{X_1|X_2}} [\ X_1|X_2 = x_2 \ ] = \int_{-\infty}^{\infty} x_1 f_{X_1|X_2}(x_1|x_2) \ dx_1 = \int_0^{x_2} x_1 \frac{1}{x_2} \ dx_1 = \frac{x_2}{2}$$

and hence by the law of iterated expectation

$$\begin{aligned} \mathrm{E}_{f_{X_1}} [\ X_1 \ ] &= \mathrm{E}_{f_{X_2}} \left[\ \mathrm{E}_{f_{X_1|X_2}} [\ X_1|X_2 = x_2 \ ] \ \right] \\ &= \int_{-\infty}^{\infty} \mathrm{E}_{f_{X_1|X_2}} [\ X_1|X_2 = x_2 \ ] \ f_{X_2}(x_2) dx_2 = \int_0^1 \frac{x_2}{2} \ 6x_2(1 - x_2) dx_2 = \frac{1}{4} \end{aligned}$$

EXAMPLE 6 Let X_1, X_2 be continuous random variables with joint density f_{X_1,X_2} and let random variable Y be defined by $Y = g(X_1, X_2)$. To calculate the pdf of Y we could use the multivariate transformation theorem after defining another (dummy) variable Z as some function of X_1 and X_2, and consider the joint transformation $(X_1, X_2) \longrightarrow (Y, Z)$.

As a special case of the Theorem, consider defining $Z = X_1$. We have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,Z}(y,z)\ dz = \int_{-\infty}^{\infty} f_{Y|Z}(y|z) f_Z(z)\ dz = \int_{-\infty}^{\infty} f_{Y|X_1}(y|x_1) f_{X_1}(x_1)\ dx_1$$

as $f_{Y,Z}(y,z) = f_{Y|Z}(y|z)f_Z(z)$ by the chain rule for densities; $f_{Y|X_1}(y|x_1)$ is a univariate (conditional) pdf for Y given $X_1 = x_1$.

Now, **given** that $X_1 = x_1$, we have that $Y = g(x_1, X_2)$, that is, Y is a transformation of X_2 only. Hence the conditional pdf $f_{Y|X_1}(y|x_1)$ can be derived using single variable (rather than multivariate) transformation techniques. Specifically, if $Y = g(x_1, X_2)$ is a 1-1 transformation from X_2 to Y, then the inverse transformation $X_2 = g^{-1}(x_1, Y)$ is well defined, and by the transformation theorem

$$f_{Y|X_1}(y|x_1) = f_{X_2|X_1}(g^{-1}(x_1,y))J(y;x_1) = f_{X_2|X_1}(\ g^{-1}(x_1,y)\ |\ x_1) \left| \frac{\partial}{\partial t} \left\{ g^{-1}(x_1,t) \right\}_{t=y} \right|$$

and hence

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_2|X_1}(\ g^{-1}(x_1,y)\ |\ x_1) \left| \frac{\partial}{\partial t} \left\{ g^{-1}(x_1,t) \right\}_{t=y} \right| f_{X_1}(x_1)\ dx_1$$

For example, if $Y = X_1 X_2$, then $X_2 = Y/X_1$, and hence

$$\left| \frac{\partial}{\partial t} \left\{ g^{-1}(x_1,t) \right\}_{t=y} \right| = \left| \frac{\partial}{\partial t} \left\{ \frac{t}{x_1} \right\}_{t=y} \right| = |x_1|^{-1}$$

so

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_2|X_1}(\ y/x_1\ |\ x_1)\, |x_1|^{-1}\, f_{X_1}(x_1)\ dx_1.$$

The conditional density $f_{X_2|X_1}$ and/or the marginal density f_{X_1} may be zero on parts of the range of the integral.

Alternatively, the cdf of Y is given by

$$F_Y(y) = \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ g(X_1, X_2) \leq y\ ] = \int_{A_y}\!\!\int f_{X_1,X_2}(x_1,x_2)\ dx_2 dx_1$$

where $A_y = \{\ (x_1, x_2)\ : g(x_1, x_2) \leq y\ \}$ so the cdf can be calculated by carefully identifying and intergrating over the set A_y.

EXAMPLE 7 Let X_1, X_2 be random variables with joint density f_{X_1,X_2} and let $g(X_1)$. Then

$$\begin{aligned}
\mathrm{E}_{f_{X_1,X_2}}[\ g(X_1)\ ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1) f_{X_1,X_2}(x_1,x_2) dx_1 dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1) f_{X_1|X_2}(x_1|x_2) f_{X_2}(x_2)\ dx_1 dx_2 \\
&= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} g(x_1) f_{X_1|X_2}(x_1|x_2)\ dx_1 \right\} f_{X_2}(x_2)\ dx_2 \\
&= \mathrm{E}_{f_{X_2}} \left[\ \mathrm{E}_{f_{X_1|X_2}}[\ g(X_1)\ |\ X_2 = x_2\ ]\ \right] = \mathrm{E}_{f_{X_1}}[\ g(X_1)\ ]
\end{aligned}$$

by the law of iterated expectation.

EXAMPLE 8 Let X_1, X_2 be continuous random variables with joint pdf given by

$$f_{X_1,X_2}(x_1,x_2) = x_1 e^{-(x_1+x_2)} \qquad x_1, x_2 > 0$$

and zero otherwise. Let $Y = X_1 + X_2$. Then by the Convolution Theorem,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, y-x_1)\ dx_1 = \int_0^y x_1 e^{-(x_1+(y-x_1))}\ dx_1 = \frac{y^2}{2}e^{-y} \qquad y > 0$$

and zero otherwise.

Note that the integral range is 0 to y as the joint density f_{X_1,X_2} is only non-zero when both its arguments are positive, that is, when $x_1 > 0$ and $y - x_1 > 0$ for fixed y, or when $0 < x_1 < y$.

It is straightforward to check that this density is a valid pdf.

EXAMPLE 9 Let X_1, X_2 be continuous random variables with joint pdf given by

$$f_{X_1,X_2}(x_1,x_2) = 2(x_1+x_2) \qquad 0 \le x_1 \le x_2 \le 1$$

and zero otherwise. Let $Y = X_1 + X_2$. Then by the Convolution Theorem,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, y-x_1)\ dx_1 = \begin{cases} \displaystyle\int_0^{y/2} 2y\ dx_1 & 0 \le y \le 1 \\[2ex] \displaystyle\int_{y-1}^{y/2} 2y\ dx_1 & 1 \le y \le 2 \end{cases}$$

and zero otherwise, as $f_{X_1,X_2}(x_1, y-x_1) = 2y$ when both x_1 and $y - x_1$ lie in the interval $[0,1]$ with $x_1 \le y - x_1$ for fixed y, and zero otherwise. Clearly Y takes values on $\mathbb{Y} = [0,2]$; for $0 \le y \le 1$, the constraints $0 \le x_1 \le y - x_1 \le 1$ imply that $0 \le x_1 \le y/2$ for fixed y, whereas if $1 \le y \le 2$ the constraints imply $1 - y \le x_1 \le y/2$.

Hence

$$f_Y(y) = \begin{cases} y^2 & 0 \le y \le 1 \\[2ex] y(2-y) & 1 \le y \le 2 \end{cases}$$

It is straightforward to check that this density is a valid pdf.

[Hint: sketch the region of (X_1, Y) space on which the joint density $f_{X_1,X_2}(x_1, y-x_1)$ is positive; this region is the triangle with corners $(0,0), (1,2), (0,1)$. The consider integration over x_1 for fixed y in the two ranges].

COVARIANCE CALCULATIONS If X_1 and X_2 are continuous random variables with joint mass function/pdf f_{X_1,X_2}, then the covariance of X_1 and X_2 is is defined by

$$\begin{aligned}
\mathrm{Cov}_{f_{X_1,X_2}}[X_1, X_2] &= \mathrm{E}_{f_{X_1,X_2}}[(X_1-\mu_1)(X_2-\mu_2)] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-\mu_1)(x_2-\mu_2)\ f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[x_1x_2 - x_1\mu_2 - x_2\mu_1 + \mu_1\mu_2]\ f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1x_2 f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 - \mu_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1 f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 \\
&\quad - \mu_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_2 f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 + \mu_1\mu_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 \\
&= \mathrm{E}_{f_{X_1,X_2}}[\,X_1X_2\,] - \mu_2\mathrm{E}_{f_{X_1}}[\,X_1\,] - \mu_1\mathrm{E}_{f_{X_2}}[\,X_2\,] + \mu_1\mu_2 \\
&= \mathrm{E}_{f_{X_1,X_2}}[\,X_1X_2\,] - \mu_1\mu_2
\end{aligned}$$

where $\mu_i = \mathrm{E}_{f_{X_i}}[X_i]$ is the marginal expectation of X_i, for $i = 1, 2$

The proof of the results for the expectation and variance of sums of random variables proceed similarly. If $Y = X_1 + X_2$, then

$$\begin{aligned}
\mathrm{E}_{f_Y}[\,Y\,] &= \mathrm{E}_{f_{X_1,X_2}}[\,X_1+X_2\,] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1+x_2)f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1 f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_2 f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 \\
&= \mathrm{E}_{f_{X_1}}[\,X_1\,] + \mathrm{E}_{f_{X_2}}[\,X_2\,] \\
\mathrm{Var}_{f_Y}[\,Y\,] &= \mathrm{Var}_{f_{X_1,X_2}}[\,X_1+X_2\,] = \mathrm{E}_{f_{X_1,X_2}}\left[\,(X_1+X_2-(\mu_1+\mu_2))^2\,\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1+x_2-\mu_1-\mu_2)^2\ f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(x_1-\mu_1)^2+(x_2-\mu_2)^2+2(x_1-\mu_1)(x_2-\mu_2)\right]\ f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-\mu_1)^2 f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_2-\mu_2)^2 f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 \\
&\quad + 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-\mu_1)(x_2-\mu_2)\ f_{X_1,X_2}(x_1,x_2)\ dx_1dx_2 \\
&= \mathrm{Var}_{f_{X_1}}[\,X_1\,] + \mathrm{Var}_{f_{X_2}}[\,X_2\,] + 2\mathrm{Cov}_{f_{X_1,X_2}}[\,X_1,X_2\,]
\end{aligned}$$

and the result for the sum of n variables follows similarly, or by induction.

EXAMPLE 10 Let X_1, X_2 be continuous random variables with joint pdf given by

$$f_{X_1,X_2}(x_1,x_2) = c \qquad 0 < x_1 < 1, x_1 < x_2 < x_1 + 1$$

and zero otherwise. To calculate c, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\ dx_2 dx_1 = \int_0^1 \int_{x_1}^{x_1+1} c\ dx_2 dx_1 = \int_0^1 c\,[x_2]_{x_1}^{x_1+1}\ dx_1 = \int_0^1 c\ dx_2 = c$$

so $c = 1$. The marginal pdf of X_1 is given by

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\ dx_2 = \int_{x_1}^{x_1+1} 1\ dx_2 = 1 \qquad 0 < x_1 < 1$$

and zero otherwise, and the marginal pdf for X_2 is given by

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\ dx_1 = \begin{cases} \displaystyle\int_0^{x_2} 1\ dx_1 & = x_2 \qquad 0 < x_2 < 1 \\ \displaystyle\int_{x_2-1}^{1} 1\ dx_1 & = 2 - x_2 \qquad 1 \le x_2 < 2 \end{cases}$$

and zero otherwise. Hence

$$\mathrm{E}_{f_{X_1}}[\ X_1\ ] = \int_{-\infty}^{\infty} x_1 f_{X_1}(x_1)\ dx_1 = \int_0^1 x_1\ dx_1 = \frac{1}{2}$$

$$\mathrm{Var}_{f_{X_1}}[\ X_1\ ] = \int_{-\infty}^{\infty} x_1^2 f_{X_1}(x_1)\ dx_1 - \left\{\mathrm{E}_{f_{X_1}}[\ X_1\ ]\right\}^2 = \int_0^1 x_1^2\ dx_1 - \frac{1}{4} = \frac{1}{12}$$

$$\begin{aligned} \mathrm{E}_{f_{X_2}}[\ X_2\ ] &= \int_{-\infty}^{\infty} x_2 f_{X_2}(x_2)\ dx_2 = \int_0^1 x_2^2\ dx_2 + \int_1^2 x_2(2-x_2)\ dx_2 \\ &= \tfrac{1}{3} - \left(1 - \tfrac{1}{3}\right) + \left(4 - \tfrac{8}{3}\right) = 1 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}_{f_{X_2}}[\ X_2\ ] &= \int_{-\infty}^{\infty} x_2^2 f_{X_2}(x_2)\ dx_2 - \left\{\mathrm{E}_{f_{X_2}}[\ X_2\ ]\right\}^2 = \int_0^1 x_2^2 x_2\ dx_2 + \int_1^2 x_2^2(2-x_2)\ dx_2 - 1 \\ &= \tfrac{1}{4} - \left(\tfrac{2}{3} - \tfrac{1}{4}\right) + \left(\tfrac{16}{3} - 4\right) - 1 = \tfrac{1}{6} \end{aligned}$$

The covariance and correlation of X_1 and X_2 are then given by

$$\begin{aligned} \mathrm{Cov}_{f_{X_1,X_2}}[\ X_1, X_2\ ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1,x_2)\ dx_2 dx_1 - \mathrm{E}_{f_{X_1}}[\ X_1\ ]\ \mathrm{E}_{f_{X_2}}[\ X_2\ ] \\ &= \int_0^1 \int_{x_1}^{x_1+1} x_1 x_2\ dx_2 dx_1\ -\ \frac{1}{2}.1 = \int_0^1 x_1 \left[\frac{x_2}{2}\right]_{x_1}^{x_1+1}\ dx_1\ -\ \frac{1}{2} \\ &= \int_0^1 \left(x_1^2 + \frac{x_1}{2}\right)\ dx_1\ -\ \frac{1}{2} = \left[\frac{x_1^3}{3} + \frac{x_1^2}{4}\right]_0^1 - \frac{1}{2} = \frac{7}{12} - \frac{1}{2} = \frac{1}{12} \end{aligned}$$

$$\mathrm{Corr}_{f_{X_1,X_2}}[\ X_1, X_2\ ] = \frac{\mathrm{Cov}_{f_{X_1,X_2}}[\ X_1, X_2\ ]}{\sqrt{\mathrm{Var}_{f_{X_1}}[\ X_1\ ]\ \mathrm{Var}_{f_{X_2}}[\ X_2\ ]}} = \frac{1/12}{\sqrt{1/12}\sqrt{1/6}} = \frac{1}{\sqrt{2}}$$