M2S1 - ASSESSED COURSEWORK 3

To be handed in no later than Thursday, 12th December
For this coursework, MAPLE may NOT be used.

You may quote without proof any results from the Formula Sheet

1. (a) Let $U \sim Uniform(0,1)$ be a continuous random variable so that

$$f_U(u) = 1 \qquad 0 < u < 1$$

and zero otherwise, and let random variable X_1 be defined in terms of U by

$$X_1 = \begin{cases} \dfrac{1}{\lambda}\log 2U & 0 < U \leq \dfrac{1}{2} \\[2ex] -\dfrac{1}{\lambda}\log(2-2U) & \dfrac{1}{2} < U < 1 \end{cases}$$

for some parameter $\lambda > 0$.

(i) Find the pdf of X_1

[4 MARKS]

[Consider the transformations on the ranges $0 < U \leq \frac{1}{2}$ and $\frac{1}{2} \leq U < 1$ separately]

(ii) Find the mgf of X_1

[6 MARKS]

(b) Suppose that the joint pdf of two continuous variables X_2 and Y is specified as

$$f_{X_2,Y}(x,y) = f_{X_2|Y}(x|y)f_Y(y)$$

where

$$X_2|Y = y \sim Normal\,(0,y) \qquad\qquad Y \sim Exponential\,(\gamma)$$

so that

$$\begin{aligned} f_{X_2|Y}(x|y) &= \left(\frac{1}{2\pi y}\right)^{1/2} \exp\left\{-\frac{x^2}{2y}\right\} \qquad x \in \mathbb{R} \\ f_Y(y) &= \gamma \exp\{-\gamma y\} \qquad y > 0 \end{aligned}$$

for parameter $\gamma > 0$.

(i) Find the (marginal) mgf of X_2

[4 MARKS]

(ii) Deduce the marginal pdf of X_2

[4 MARKS]

(iii) Find the variance of X_2

[2 MARKS]

2. (a) Suppose that random variables $X_1, X_2, ...X_n$ are independently and identically distributed $Poisson(\lambda)$ random variables.

(i) Find the distribution of

$$T_n = \sum_{i=1}^{n} X_i$$

[2 MARKS]

(ii) By considering a suitably **standardized** variable, show that, as $n \to \infty$, the distribution of T_n can be approximated

$$F_{T_n}(t) \approx \Phi\left(\frac{t-\mu}{\sigma}\right)$$

for parameters μ and σ^2 to be identified.

[4 MARKS]

(b) To discover whether a new drug treatment produces a higher recovery rate than a placebo control, a clinical trial was undertaken on a cohort of patients, on the basis of age and medical history, were deemed likely to respond similarly in the trial, and who were then randomly allocated to each of trial arms.

Suppose that, of n_0 patients allocated to the placebo control, x_0 were deemed to have recovered, whereas $n_0 - x_0$ patients did not. Similarly, in the treatment group comprising n_1 patients, x_1 recovered and $n_1 - x_1$ did not. All patients are deemed to respond independently of each other. Evidence to support the use of the new drug is being sought: it is hypothesized, that in the control group the probability that a patient recovers is θ_0, whereas in the treatment group, this probability is θ_1

(i) Let X_0 and X_1 denote the random variables representing the numbers of patients that recovered in each of the control and treatment groups respectively. Show that, for $i = 0, 1$, as $n_i \to \infty$, the random variable

$$Z_i = \frac{X_i - n_i\theta_i}{\sqrt{n_i\theta_i(1-\theta_i)}}$$

is approximately $Normal(0, 1)$ distributed.

[4 MARKS]

(ii) Find the approximate distribution of random variable $Z_1 - Z_0$ for large n_0, n_1

[2 MARKS]

(iii) Find approximations to the distributions of random variables

$$\begin{aligned} Q_0 &= Z_0^2, \\ Q_1 &= Z_1^2 \\ Q &= Q_0 + Q_1 \end{aligned}$$

[4 MARKS]

(iv) It is now hypothesized that $\theta_0 = \theta_1$. Under this assumption, find the distribution of random variable

$$X = X_0 + X_1$$

and a Normal approximation to the distribution of X.

[4 MARKS]