M2S1 - EXERCISES 7: SOLUTIONS

1. Key is to find the i.i.d random variables $X_1, ..., X_n$ such that

$$X = \sum_{i=1}^{n} X_i$$

and then to use the Central Limit Theorem result for large n

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} \to Z \sim Normal(0,1) \qquad \text{so that } X = \sum_{i=1}^{n} X_i \sim Normal(n\mu, n\sigma^2) \text{ approximately}$$

where $\mu = \mathrm{E}_{f_X}[X_i]$ and $\sigma^2 = \mathrm{Var}_{f_X}[X_i]$

(i) $X \sim Binomial(n,\theta) \implies X = \sum_{i=1}^{n} X_i$ where $X_i \sim Bernoulli(\theta)$ so that $\mu = \mathrm{E}_{f_X}[X_i] = \theta$ and $\sigma^2 = \mathrm{Var}_{f_X}[X_i] = \theta(1-\theta)$ and hence

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\theta}{\sqrt{n\theta(1-\theta)}} \sim Normal(0,1) \implies X \sim Normal(n\theta, n\theta(1-\theta)) \text{ approximately}$$

(ii) $X \sim Poisson(\lambda) \implies X = \sum_{i=1}^{n} X_i$ where $X_i \sim Poisson\,(\lambda/n)$ so that $\mu = \mathrm{E}_{f_X}[X_i] = \lambda/n$ and $\sigma^2 = \mathrm{Var}_{f_X}[X_i] = \lambda/n$ and hence

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\dfrac{\lambda}{n}}{\sqrt{n\,(\lambda/n)}} = \frac{\sum_{i=1}^{n} X_i - \lambda}{\sqrt{\lambda}} \sim Normal(0,1) \implies X \sim Normal(\lambda,\lambda) \text{ approximately}$$

Note that this uses the result that the sum of independent Poisson variables also has a Poisson distribution (proved using mgfs), and also note that this is in agreement with the mgf limit result for the "standardized" Poisson example given in lectures.

(iii) $X \sim NegBinomial(n,\theta) \implies X = \sum_{i=1}^{n} X_i$ where $X_i \sim Geometric(\theta)$ so that $\mu = \mathrm{E}_{f_X}[X_i] = 1/\theta$ and $\sigma^2 = \mathrm{Var}_{f_X}[X_i] = (1-\theta)/\theta^2$ and hence

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\dfrac{1}{\theta}}{\sqrt{n\left((1-\theta)/\theta^2\right)}} \sim Normal(0,1) \implies X \sim Normal\left(\frac{n}{\theta}, \frac{n(1-\theta)}{\theta^2}\right) \text{ approximately}$$

(iv) $X \sim Gamma(\alpha,\beta) \implies X = \sum_{i=1}^{n} X_i$ where $X_i \sim Gamma\left(\dfrac{\alpha}{n},\beta\right)$ so that $\mu = \mathrm{E}_{f_X}[X_i] = \dfrac{\alpha}{n\beta}$ and $\sigma^2 = \mathrm{Var}_{f_X}[X_i] = \dfrac{\alpha}{n\beta^2}$ and hence

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\dfrac{\alpha}{n\beta}}{\sqrt{n\alpha/\left(n\beta^2\right)}} = \frac{\sum_{i=1}^{n} X_i - \dfrac{\alpha}{\beta}}{\sqrt{\alpha/\beta^2}} \sim Normal(0,1) \implies X \sim Normal\left(\frac{\alpha}{\beta}, \frac{\alpha}{\beta^2}\right) \text{ approximately}$$

This is essentially the mgf limit result for the "standardized" Gamma example given in lectures in the special case $\beta = 1$.

2. $Y_n = \max\{X_1, ..., X_n\}$ so in the limit as $n \to \infty$ we have the limit for *fixed* y as

$$F_{Y_n}(y) = \{F_X(y)\}^n = y^n \to \begin{cases} 0 & y < 1 \\ 1 & y \geq 1 \end{cases}$$

that is, a step function with single step of size 1 at $y = 1$. Hence the limiting random variable Y is a discrete variable with $P[Y = 1] = 1$, that is, the limiting distribution is *degenerate* at 1. For $Z_n = \min\{X_1, ..., X_n\}$ so in the limit as $n \to \infty$ we have the limit for *fixed* z as

$$F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n = 1 - (1-z)^n \to \begin{cases} 0 & z \leq 0 \\ 1 & z > 0 \end{cases}$$

that is, a step function with single step of size 1 at $z = 0$. Hence the limiting random variable Z is a discrete variable with $P[Z = 0] = 1$, that is, the limiting distribution is *degenerate* at 0. Note here that the limiting function is **not** a cdf as it is not **right-continuous**, but that the limiting distribution does still exist - the ordinary definition of convergence in distribution only refers to pointwise convergence **at points of continuity of the limit function**, and here is limit function is not continuous at zero.

Note that these results are intuitively reasonable as, as the sample size gets increasingly large, we will obtain a random variable arbitrarily close to each end of the range. Note also that these results describe *convergence in distribution*, but also we have for $1 > \varepsilon > 0$

$$\begin{array}{l} \mathrm{P}[|Y_n - 1| < \varepsilon] = \mathrm{P}[1 - Y_n < \varepsilon] = \mathrm{P}[1 - \varepsilon < Y_n] = 1 - \mathrm{P}[Y_n < 1 - \varepsilon] = 1 - \varepsilon^n \to 1 \\ \mathrm{P}[|Z_n - 0| < \varepsilon] = \mathrm{P}[Z_n < \varepsilon] = 1 - (1 - \varepsilon)^n \to 1 \end{array} \quad \text{as } n \to \infty$$

so we also have *convergence in probability* of Y_n to 1 and of Z_n to 0

3. $Z_n = \min\{X_1, ..., X_n\}$ so

$$F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n = 1 - \left(1 - \left(1 - \frac{1}{z}\right)\right)^n = 1 - \frac{1}{z^n} \qquad z > 1$$

and so, in the limit as $n \to \infty$ we have the limit for *fixed* z as

$$F_{Z_n}(z) \to \begin{cases} 0 & z \leq 1 \\ 1 & z > 1 \end{cases}$$

that is, a step function with single step of size 1 at $z = 1$. Hence the limiting random variable Z is a discrete variable with

$$P[Z = 1] = 1$$

that is, the limiting distribution is *degenerate* at 1. Again, the limiting function is not a cdf as it not right continuous, but this does not affect our conclusion, as the limit function is not continuous at 1.

Now if $U_n = Z_n^n$, we have from first principles that for $u > 1$

$$F_{U_n}(u) = \mathrm{P}[U_n \leq u] = \mathrm{P}[Z_n^n \leq u] = \mathrm{P}\left[Z_n \leq u^{1/n}\right] = 1 - \frac{1}{\left(u^{1/n}\right)^n} = 1 - \frac{1}{u}$$

which is a valid cdf, but which does not depend on n. Hence the limiting distribution of U_n is precisely

$$F_U(u) = 1 - \frac{1}{u} \qquad u > 1$$

For $u \leq 1$, $F_{U_n}(u) = 0$ for all n, so clearly $F_{U_n}(u) \longrightarrow 0$ for u in this range. Hence the limiting distribution function is continuous at $u = 1$ (indeed, at all u).

4. $Y_n = \max\{X_1, ..., X_n\}$ so

$$F_{Y_n}(y) = \{F_X(y)\}^n = \left(\frac{1}{1+e^{-y}}\right)^n \qquad y \in \mathbb{R}$$

and so, in the limit as $n \to \infty$ we have the limit for *fixed* y as

$$F_{Y_n}(y) \to 0 \qquad \text{for all } y$$

Hence there is *no limiting distribution.*

If $U_n = Y_n - \log n$, we have from first principles that for $u > -\log n$

$$F_{U_n}(u) = \mathrm{P}[U_n \le u] = \mathrm{P}[Y_n - \log n \le u] = \mathrm{P}[Y_n \le u + \log n] = F_{Y_n}(u + \log n) = \left(\frac{1}{1+e^{-u-\log n}}\right)^n$$

so that

$$F_{U_n}(u) = \left(\frac{1}{1+\dfrac{e^{-u}}{n}}\right)^n = \left(1+\frac{e^{-u}}{n}\right)^{-n} \to \exp\left\{-e^{-u}\right\} \qquad \text{as } n \to \infty$$

which is a valid cdf. Hence the limiting distribution is

$$F_U(u) = \exp\left\{-e^{-u}\right\} \qquad u \in \mathbb{R}$$

5. $Y_n = \max\{X_1, ..., X_n\}$ so

$$F_{Y_n}(y) = \{F_X(y)\}^n = \left(\frac{\lambda y}{1+\lambda y}\right)^n \qquad y > 0$$

and so, in the limit as $n \to \infty$ we have the limit for *fixed* y as

$$F_{Y_n}(y) \to 0 \qquad \text{for all } y$$

Hence there is *no limiting distribution.*

$Z_n = \min\{X_1, ..., X_n\}$ so in the limit as $n \to \infty$ we have the limit for *fixed* $z > 0$ as

$$F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n = 1 - \left(1 - \left(1 - \frac{1}{1+\lambda z}\right)\right)^n = 1 - \frac{1}{(1+\lambda z)^n} \to \begin{cases} 0 & z \le 0 \\ 1 & z > 0 \end{cases}$$

that is, a step function with single step of size 1 at $z = 0$. Hence the limiting random variable Z is a discrete variable with $\mathrm{P}[Z = 0] = 1$ that is, the limiting distribution is *degenerate* at 0. Again, the limiting function is not a cdf as it not right continuous, but this does not affect our conclusion, as the limit function is not continuous at 0.

If $U_n = Y_n/n$, we have from first principles that for $u > 0$

$$F_{U_n}(u) = \mathrm{P}[U_n \le u] = \mathrm{P}[Y_n/n \le u] = \mathrm{P}[Y_n \le nu] = F_{Y_n}(nu) = \left(\frac{\lambda nu}{1+\lambda nu}\right)^n$$

so that

$$F_{U_n}(u) = \left(\frac{\lambda nu}{1+\lambda nu}\right)^n = \left(1+\frac{1}{n\lambda u}\right)^{-n} \to \exp\left\{-\frac{1}{\lambda u}\right\} \qquad \text{as } n \to \infty$$

which is a valid cdf. Hence the limiting distribution is

$$F_U(u) = \exp\left\{-\frac{1}{\lambda u}\right\} \qquad u > 0$$

If $V_n = nZ_n$, we have from first principles that for $u > 0$

$$F_{V_n}(v) = \mathrm{P}\left[V_n \leq v\right] = \mathrm{P}\left[nZ_n \leq v\right] = \mathrm{P}\left[Z_n \leq v/n\right] = F_{Z_n}(v/n) = 1 - \left(\frac{1}{1+\dfrac{\lambda v}{n}}\right)^n$$

so that

$$F_{V_n}(v) = 1 - \left(1+\frac{\lambda v}{n}\right)^{-n} = 1 - \left(1+\frac{\lambda v}{n}\right)^{-n} \to 1 - \exp\left\{-\lambda v\right\} \qquad \text{as } n \to \infty$$

which is a valid cdf. Hence the limiting distribution is

$$F_V(v) = 1 - \exp\left\{-\lambda v\right\} \qquad v > 0$$

Hence the limiting random variable $V \sim \mathit{Exponential}(\lambda)$.

$Y_n = \max\left\{X_1, ..., X_n\right\}$ so

$$F_{Y_n}(y) = \left\{F_X(y)\right\}^n = (1 - e^{-\lambda y})^n \qquad y > 0$$

6. $X_i \sim \mathit{Poisson}(\lambda)$ so $\sum_{i=1}^{n} X_i \sim \mathit{Poisson}(n\lambda)$ by mgfs and hence (by Q1 result) using the Central Limit Theorem,

$$\sum_{i=1}^{n} X_i \sim \mathit{Normal}(n\lambda, n\lambda) \qquad \text{approximately}$$

and hence

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathit{Normal}\left(\lambda, \frac{\lambda}{n}\right) \qquad \text{approximately}$$

and hence, for $\varepsilon > 0$

$$\mathrm{P}\left[\left|\overline{X} - \lambda\right| < \varepsilon\right] = \mathrm{P}\left[\lambda - \varepsilon < \overline{X} < \lambda + \varepsilon\right] \approx \Phi\left(\frac{\varepsilon}{\sqrt{\lambda/n}}\right) - \Phi\left(\frac{-\varepsilon}{\sqrt{\lambda/n}}\right) \to 1$$

as $n \to \infty$. Hence, $\overline{X}$ *converges in probability* to λ

$$\overline{X} \xrightarrow{p} \lambda$$

Now, if $T_n = \exp\left\{-M_n\right\}$, then for $\varepsilon > 0$ we have

$$P\left[\left|T_n - e^{-\lambda}\right| < \varepsilon\right] = P\left[e^{-\lambda} - \varepsilon < T_n < e^{-\lambda} + \varepsilon\right] = P\left[-\log(e^{-\lambda} + \varepsilon) < M_n < -\log(e^{-\lambda} - \varepsilon)\right]$$

and hence

$$P\left[\left|T_n - e^{-\lambda}\right| < \varepsilon\right] \approx \Phi\left(\frac{-\log(e^{-\lambda} - \varepsilon) - \lambda}{\sqrt{\lambda/n}}\right) - \Phi\left(\frac{-\log(e^{-\lambda} + \varepsilon) - \lambda}{\sqrt{\lambda/n}}\right) \to 1$$

as $n \to \infty$. Hence, T_n *converges in probability* to $e^{-\lambda}$.