M2S1 - EXERCISES 3: SOLUTIONS

1. Clearly F_X is continuous, and if $c=1$,

$$\lim_{x\longrightarrow -\infty} F_X(x) = 0 \qquad \lim_{x\longrightarrow \infty} F_X(x) = 1$$

so F_X is a cdf.. To find the pdf, differentiate F_X;

$$f_X(x) = \frac{d}{dt}\left\{F_X(t)\right\}_{t=x} = \frac{d}{dt}\left\{\exp\left\{-e^{-\lambda t}\right\}\right\}_{t=x} = \lambda \exp\left\{-\lambda x - e^{-\lambda x}\right\} \qquad x \in \mathbb{R}.$$

If $f_X(x) = cg(x)$ is a pdf, then the corresponding cdf F_X is defined by

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt \ = \begin{cases} \displaystyle\int_{-\infty}^{x} -\frac{ct}{(1+t^2)^2}\ dt \\ \displaystyle\int_{-\infty}^{0} -\frac{ct}{(1+t^2)^2}\ dt + \int_0^x \frac{ct}{(1+t^2)^2}\ dt \end{cases}$$

$$= \begin{cases} \left[\dfrac{c}{2}\dfrac{1}{1+t^2}\right]_{-\infty}^{x} & x \le 0 \\ \dfrac{c}{2} + \left[-\dfrac{c}{2}\dfrac{1}{1+t^2}\right]_0^x & x > 0 \end{cases} = \begin{cases} \dfrac{c}{2(1+x^2)} & x \le 0 \\ \dfrac{c(1+2x^2)}{2(1+x^2)} & x > 0 \end{cases}$$

and hence $c=1$, as we must have $\displaystyle\lim_{x\longrightarrow\infty} F_X(x) = 1$

$\mathrm{E}_{f_X}[\ X\ ] = 0$ as f_X is symmetric about 0, and the expectation integral is *finite*. We know that

$$\int_{-\infty}^{\infty} x f_X(x)\ dx = \int_{-\infty}^{0} \frac{-x^2}{(1+x^2)^2}\ dx + \int_0^{\infty} \frac{x^2}{(1+x^2)^2}\ dx = 0$$

as the integrands in these integrals behave like $1/x^2$ as x becomes large, and hence the integrals are finite, and cancel as they are equal and opposite in sign.

2.

$$\begin{aligned} E_{f_X}[\ X\ ] &= \int_0^{\infty} x f_X(x)\ dx = \int_0^{\infty} \left\{\int_0^x dy\right\} f_X(x)\ dx = \int_0^{\infty}\left\{\int_y^{\infty} f_X(x)\ dx\right\} dy \\ &= \int_0^{\infty} (1 - F_X(y))\ dy \equiv \int_0^{\infty} (1-F_X(x))\ dx \end{aligned}$$

$$\begin{aligned} E_{f_X}[\ X^r\ ] &= \int_0^{\infty} x^r f_X(x)\ dx = \int_0^{\infty}\left\{\int_0^x r y^{r-1}\ dy\right\} f_X(x)\ dx = \int_0^{\infty}\left\{\int_y^{\infty} f_X(x)\ dx\right\} r y^{r-1}\ dy \\ &= \int_0^{\infty} (1-F_X(y)) r y^{r-1}\ dy \equiv \int_0^{\infty} r x^{r-1}(1-F_X(x))\ dx \end{aligned}$$

Note: the exchange of order of integration is valid if we know that the expectation integral is finite. The result is also holds in the discrete case with integrals replaced by summations. The important thing is to remember the trick of introducing a second integral involving dummy variable y. The rest of the result follows after careful manipulation of the double integral.

Now, for a random variable that takes values on $\mathbb{R}$, we split the integral into two at the origin and proceed as above, as follows.

$$\begin{aligned}
E_{f_X}[\, X^r\,] &= \int_{-\infty}^{\infty} x^r f_X(x)\ dx = \int_{-\infty}^{0} x^r f_X(x)\ dx + \int_0^{\infty} x^r f_X(x)\ dx \\
&= \int_{-\infty}^{0} \left\{ \int_0^x r y^{r-1}\ dy \right\}\ f_X(x)\ dx + \int_0^{\infty} r x^{r-1}(1 - F_X(x))\ dx \\
&= \int_{-\infty}^{0} \left\{ -\int_x^0 r y^{r-1}\ dy \right\}\ f_X(x)\ dx + \int_0^{\infty} (1 - F_X(y)) r y^{r-1}\ dy \\
&= -\int_{-\infty}^{0} r y^{r-1} \left\{ \int_{-\infty}^{y} f_X(x)\ dx \right\}\ dy + \int_0^{\infty} (1 - F_X(y)) r y^{r-1}\ dy \\
&= -\int_{-\infty}^{0} r y^{r-1} F_X(y)\ dy + \int_0^{\infty} (1 - F_X(y)) r y^{r-1}\ dy
\end{aligned}$$

3. We have that

$$\begin{aligned}
\mathrm{E}_{f_2}[\, X_2^r\,] &= \int_0^{\infty} x^r f_2(x)\ dx = \int_0^{\infty} x^r \left[1 + \sin(2\pi \log x)\right] f_1(x)\ dx \\
&= \int_0^{\infty} x^r f_1(x)\ dx + \int_0^{\infty} x^r \sin(2\pi \log x) f_1(x)\ dx \\
&= \mathrm{E}_{f_1}[\, X_1^r\,] + \int_0^{\infty} x^r \sin(2\pi \log x) c x^{-1} \exp\left\{ -\frac{(\log x)^2}{2} \right\}\ dx \\
&= \mathrm{E}_{f_1}[\, X_1^r\,] + c \int_{-\infty}^{\infty} e^{rt} \sin(2\pi t) \exp\left\{ -\frac{t^2}{2} \right\}\ dt \qquad \text{(putting } t = \log x\text{)} \\
&= \mathrm{E}_{f_1}[\, X_1^r\,] + c \exp\left\{ r^2/2 \right\} \int_{-\infty}^{\infty} \sin(2\pi t) \exp\left\{ -\frac{(t-r)^2}{2} \right\}\ dt \qquad \text{(completing the square in } t\text{))} \\
&= \mathrm{E}_{f_1}[\, X_1^r\,] + c \exp\left\{ r^2/2 \right\} \int_{-\infty}^{\infty} \sin(2\pi (s+r)) \exp\left\{ -\frac{s^2}{2} \right\}\ ds \qquad \text{(putting } s = t - r\text{)} \\
&= \mathrm{E}_{f_1}[\, X_1^r\,] + c \exp\left\{ r^2/2 \right\} \int_{-\infty}^{\infty} \sin(2\pi s) \exp\left\{ -\frac{s^2}{2} \right\}\ ds \ = \mathrm{E}_{f_1}[\, X_1^r\,]
\end{aligned}$$

as $\sin(2\pi(s+r)) = \sin(2\pi s)$ for $r = 1, 2, ...$, as the integrand is an integrable, odd function about zero.

The result follows after showing that the second integral is zero; it may not be obvious when you start the manipulation, but the $t = \log x$ substitution seems a natural first step - this has two advantages; first it gets rid of the awkward log terms and secondly it changes the range of integration to the whole real line leaving an integrand that looks more familiar and tractable. The next step of completing the square takes a little spotting, but also seems sensible to combine the exp terms. The remainder of the calculation is similar to the the example given in lectures; here the integral is zero as the integrand is an integrable **odd** function.

4. (i) By integration, for $x \geq 0$,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt = \int_0^x \alpha^2 t \exp\{-\alpha t\}\ dt = [-\alpha t \exp\{-\alpha t\}]_0^x + \int_0^x \alpha \exp\{-\alpha t\}\ dt$$

$$= -\alpha x \exp\{-\alpha x\} + [-\exp\{-\alpha t\}]_0^x = 1 - (1+\alpha x)\exp\{-\alpha x\}$$

Hence $\mathrm{P}[\ X \geq m\ ] = 1 - \mathrm{P}[\ X < m\ ] = 1 - F_X(m) = (1+\alpha m)\exp\{-\alpha m\}$

(ii)

$$\mathrm{E}_{f_X}[\ X\ ] = \int_{-\infty}^{\infty} x f_X(x)\ dx = \int_0^{\infty} x\alpha^2 x \exp\{-\alpha x\}\ dx = \left[-\alpha x^2 \exp\{-\alpha x\}\right]_0^{\infty} + \int_0^{\infty} 2x\alpha \exp\{-\alpha x\}\ dx$$

$$= 0 + \frac{2}{\alpha}\int_0^{\infty} x\alpha^2 \exp\{-\alpha x\}\ dx = \frac{2}{\alpha}$$

as the integrand is a pdf. Hence a change in the expectation to $2/\beta$ corresponds to a change from α to β in the pdf and cdf. Hence $\mathrm{P}[\ X \geq m\ ]$ changes to $(1+\beta m)\exp\{-\beta m\}$.

5. (a) To calculate the mgf

$$M_Z(t) = E_{f_Z}[\ e^{tZ}\ ] = \int_{-\infty}^{\infty} e^{zt}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}\ dz = e^{t^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(z-t)^2}{2}\right\}\ dz$$

$$= e^{t^2/2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{u^2}{2}\right\}\ du = e^{t^2/2}$$

completing the square in z, and then setting $u = z - t$, as the integrand is a pdf.
Now, using the transformation theorem for univariate, 1-1 transformations we have $X = \mu + \dfrac{1}{\lambda}Z \Longleftrightarrow Z = \lambda(X-\mu)$, so

$$f_X(x) = f_Z(\lambda(x-\mu))\ \lambda = \frac{\lambda}{\sqrt{2\pi}}\exp\left\{-\frac{\lambda^2}{2}(x-\mu)^2\right\} \qquad x \in \mathbb{R}$$

To calculate the mgf of X, use the expectation result given in lectures

$$M_X(t) = E_{f_Z}\left[e^{t(\mu+Z/\lambda)}\right] = e^{\mu t}M_Z(t/\lambda) = \exp\left\{\mu t + \frac{t^2}{2\lambda^2}\right\}$$

The expectation of X is

$$E_{f_X}[X] = \int_{-\infty}^{\infty} x f_X(x)\ dx = \int_{-\infty}^{\infty} x\left(\frac{\lambda^2}{2\pi}\right)^{1/2}\exp\left\{-\frac{\lambda^2}{2}(x-\mu)^2\right\}\ dx$$

$$= \int_{-\infty}^{\infty}\left(\mu + t\lambda^{-1}\right)\left(\frac{\lambda^2}{2\pi}\right)^{1/2}\exp\left\{-\frac{t^2}{2}\right\}\ \lambda^{-1}\ dt \qquad t = \lambda(x-\mu)$$

$$= \mu\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\right)^{1/2}\exp\left\{-\frac{t^2}{2}\right\}\ dt + \lambda^{-1}\int_{-\infty}^{\infty} t\left(\frac{1}{2\pi}\right)^{1/2}\exp\left\{-\frac{t^2}{2}\right\}\ dt$$

$$= \mu$$

as the first integral is 1, and the second integral is zero, as the integrand is an ODD function about zero. Hence

$$E_{f_X}[X] = \mu$$

and note that it is generally true that if a pdf is symmetric about a particular value, then that value is the expectation (if the expectation integral is finite). Alternately, could use the mgf result that says

$$E_{f_X}[X] = \frac{d}{ds}\left\{M_X(s)\right\}_{s=0} = M_X^{(1)}(0)$$

say, so that

$$E_{f_X}[X] = \frac{d}{ds}\left\{\exp\left\{\mu s + \frac{s^2}{2\lambda^2}\right\}\right\}_{s=0} = \left\{\left(\mu + \frac{s}{\lambda^2}\right)\exp\left\{\mu s + \frac{s^2}{2\lambda^2}\right\}\right\}_{s=0} = \mu$$

The expectation of $g(X) = e^X$ is

$$\begin{aligned}
E_{f_X}[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_{-\infty}^{\infty} e^x \left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\lambda^2}{2}(x-\mu)^2\right\}\, dx \\
&= \int_{-\infty}^{\infty} \exp\left\{\mu + t\lambda^{-1}\right\}\left(\frac{\lambda^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{t^2}{2}\right\}\lambda^{-1}\, dt \qquad \text{setting } t = \lambda(x-\mu) \\
&= \left(\frac{1}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} \exp\left\{\mu + t\lambda^{-1} - \frac{t^2}{2}\right\}\, dt = \left(\frac{1}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(t^2 - 2t\lambda^{-1} - 2\mu\right)\right\}\, dt
\end{aligned}$$

Completing the square in the exponent, we have

$$\left(t^2 - 2t\lambda^{-1} - 2\mu\right) = \left(t - \lambda^{-1}\right)^2 - \left(2\mu + \lambda^{-2}\right)$$

and hence

$$\begin{aligned}
E_{f_X}[g(X)] &= \left(\frac{1}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(t-\lambda^{-1}\right)^2 + \left(\mu + \frac{1}{2\lambda^2}\right)\right\}\, dt \\
&= \exp\left\{\mu + \frac{1}{2\lambda^2}\right\}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\right)^{1/2}\exp\left\{-\frac{1}{2}\left(t-\lambda^{-1}\right)^2\right\}\, dt = \exp\left\{\mu + \frac{1}{2\lambda^2}\right\}
\end{aligned}$$

as the integral is equal to 1, as it is the integral of a pdf for all choices of λ.

(b) If $Y = e^X$, so $\mathbb{Y} = R^+$, and from first principles we have

$$F_Y(y) = \mathrm{P}[Y \le y] = \mathrm{P}\left[e^X \le y\right] = \mathrm{P}[X \le \log y] = F_X(\log y)$$

so by differentiation

$$f_Y(y) = f_X(\log y)\,\frac{1}{y} \qquad y > 0$$

Note that the function $g(t) = e^t$ is a monotone increasing function, with $g^{-1}(t) = \log t$, so that we can use the transformation result directly, that is

$$f_Y(y) = f_X(g^{-1}(y))\; J(y) \qquad \text{where} \qquad J(y) = \left|\frac{d}{dt}\left\{g^{-1}(t)\right\}_{t=y}\right| = \left|\frac{d}{dt}\left\{\log t\right\}_{t=y}\right| = \frac{1}{y}$$

Hence

$$f_Y(y) = \frac{1}{y}\left(\frac{\lambda^2}{2\pi}\right)^{1/2}\exp\left\{-\frac{\lambda^2}{2}(\log y - \mu)^2\right\} \qquad y > 0.$$

For the expectation, we have from first principles

$$
\begin{aligned}
E_{f_Y}[Y] &= \int_0^\infty y f_Y(y)\,dy = \int_{-\infty}^{\infty} y\,\frac{1}{y}\left(\frac{\lambda^2}{2\pi}\right)^{1/2}\exp\left\{-\frac{\lambda}{2}(\log y-\mu)^2\right\}\,dy \\
&= \int_{-\infty}^{\infty}\left(\frac{\lambda^2}{2\pi}\right)^{1/2}\exp\left\{-\frac{\lambda^2}{2}(t-\mu)^2\right\}\,e^t\,dt \quad = \quad \exp\left\{\mu+\frac{1}{2\lambda^2}\right\}
\end{aligned}
$$

where $t=\log y$, as the integral is precisely the one carried out above. This illustrates the transformation/expectation result that, if $Y=g(X)$, then

$$E_{f_Y}[Y] = E_{f_X}[g(X)]$$

(c) If $T=Z^2$, then from first principles

$$
\begin{aligned}
F_T(t) &= P[\ T\le t\ ] = P[\ Z^2\le t\ ] = P[\ -\sqrt{t}\le Z\le\sqrt{t}\ ] \\
\Longrightarrow f_T(t) &= \frac{1}{2\sqrt{t}}\left[f_Z(\sqrt{t})+f_Z(-\sqrt{t})\right] = \frac{1}{\sqrt{2\pi}}t^{-1/2}\exp\left\{-\frac{t}{2}\right\} \qquad t>0
\end{aligned}
$$

and hence

$$
\begin{aligned}
M_T(t) = E_{f_T}[\ e^{tT}\ ] = \int_{-\infty}^{\infty} e^{tx} f_T(x)\,dx \quad &= \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi x}}\exp\left\{-\frac{x}{2}\right\}\,dx \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi x}}\exp\left\{-\frac{(1-2t)x}{2}\right\}\,dx \\
&= \left(\frac{1}{1-2t}\right)^{1/2}\int_0^\infty \frac{1}{\sqrt{2\pi y}}\exp\left\{-\frac{y}{2}\right\}\,dy = \left(\frac{1}{1-2t}\right)^{1/2}
\end{aligned}
$$

where $y=(1-2t)x$, as the integrand is a pdf.

6. By definition of mgfs for discrete variables, we can deduce immediately that, as

$$M_X(t) = \sum_{x=-\infty}^{\infty} e^{tx} f_X(x)$$

$P[\ X=x\ ]$ is just the coefficient of e^{tx} in the expression for M_X, and hence $P[\ X=1\ ]=1/8$, $P[\ X=2\ ]=1/4$ and $P[\ X=3\ ]=5/8$. Also, we have $E_{f_X}[\ X^r\ ]=M_X^{(r)}(0)$, so that

$$
\begin{aligned}
E_{f_X}[\ X\ ] &= M_X^{(1)}(0) = \frac{1}{8}+2\frac{1}{4}+3\frac{5}{8} = \frac{5}{2} \\
E_{f_X}[\ X^2\ ] &= M_X^{(2)}(0) = \frac{1}{8}+4\frac{1}{4}+9\frac{5}{8} = \frac{27}{4}
\end{aligned}
$$

so therefore

$$Var_{f_X}[\ X\ ] = E_{f_X}[\ X^2\ ] - \{E_{f_X}[\ X\ ]\}^2 = \frac{1}{2}$$

7. Can identify that $X \sim Binomial(n, \theta)$, but in any case,

$$
\begin{aligned}
M_X(t) &= (1-\theta+\theta e^t)^n = (1+(e^t-1)\theta)^n = \left(1+\theta\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+...\right)\right)^n \\
&= \sum_{r=0}^{n}\binom{n}{r}\theta^r\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+...\right)^r
\end{aligned}
$$

and from the mgf definition $E_{f_X}[\ X^r\ ]$ is $r!$ times the coefficient of t^r. Difficult to identify this general term, but can easily identify the coefficient of t as $n\theta = E_{f_X}[\ X\ ]$, and the coefficient of t^2 as $n\theta + n(n-1)\theta^2 = E_{f_X}[\ X^2\ ]$ etc.

8. For this pdf,

$$
\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\ dx = \int_{-2}^{\infty} e^{tx} e^{-(x+2)}\ dx = e^{-2}\int_{-2}^{\infty} e^{-(1-t)x}\ dx \\
&= \frac{e^{-2}}{1-t}\int_{-2(1-t)}^{\infty} e^{-y}\ dy \ = \ \frac{e^{-2}}{1-t}\ \left[-e^{-y}\right]_{-2(1-t)}^{\infty} \ = \ \frac{e^{-2t}}{1-t} \qquad t<1
\end{aligned}
$$

Now

$$
M_X^{(1)}(t) = \frac{e^{-2t}}{(1-t)^2}(2t-1) \qquad M_X^{(2)}(t) = \frac{e^{-2t}}{(1-t)^3}\left[1+(2t-1)^2\right]
$$

so that $M_X^{(1)}(0) = -1 = E_{f_X}[\ X\ ]$ and $M_X^{(2)}(0) = 2 = E_{f_X}[\ X^2\ ] \Longrightarrow \text{Var}_{f_X}[\ X\ ] = 1$

9. We have $K_X(t) = \log M_X(t)$, hence

$$
K_X^{(1)}(t) = \frac{d}{ds}\left\{K_X(t)\right\}_{s=t} = \frac{d}{ds}\left\{\log M_X(t)\right\}_{s=t} = \frac{M_X^{(1)}(t)}{M_X(t)} \Longrightarrow K_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = E_{f_X}[\ X\ ]
$$

as $M_X(0) = 1$. Similarly

$$
K_X^{(2)}(t) = \frac{M_X(t)M_X^{(2)}(t) - \left\{M_X^{(1)}(t)\right\}^2}{\left\{M_X(t)\right\}^2}
$$

and hence

$$
K_X^{(2)}(0) = \frac{M_X(0)M_X^{(2)}(0) - \left\{M_X^{(1)}(0)\right\}^2}{\left\{M_X(0)\right\}^2} = E_{f_X}[\ X^2\ ] - \left\{E_{f_X}[\ X\ ]\right\}^2
$$

and hence $K_X^{(2)}(0) = Var_{f_X}[\ X\ ]$