M2S1 - EXERCISES 2: SOLUTIONS

1. Need $\sum_{x=1}^{\infty} f_X(x) = 1$. Hence

$$\text{(a)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{1}{2^x} = 1 \qquad \text{(b)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{1}{x2^x} = \log 2$$

$$\text{(c)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6} \qquad \text{(d)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{2^x}{x!} = e^2 - 1$$

(a) is given by the sum of a geometric progression; (b) uses the fact that

$$\frac{1}{1-t} = 1 + t + t^2 + ... = \sum_{x=0}^{\infty} t^x \Longrightarrow -\log(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + ... = \sum_{x=1}^{\infty} \frac{t^x}{x}$$

by integrating both sides with respect to t. Hence for $t = 1/2$, we have

$$\log 2 = -\log(1 - 1/2) = \sum_{x=1}^{\infty} \frac{1}{x2^x}.$$

(c) is a "well-known" mathematical result (**you would not be expected to remember it for the examination**); (d) uses the power series expansion of e^t, evaluated at $t = 2$, that is

$$e^t = \sum_{x=0}^{\infty} \frac{t^x}{x!} \Longrightarrow e^2 = \sum_{x=0}^{\infty} \frac{2^x}{x!} = 1 + \sum_{x=1}^{\infty} \frac{2^x}{x!}$$

Clearly P[$X > 1$] = 1 − P[$X = 1$], so

$$\text{(a)} \quad \mathrm{P}[\ X > 1\ ] = \frac{1}{2} \qquad \text{(b)} \quad \mathrm{P}[\ X > 1\ ] = 1 - \frac{1}{2\log 2}$$

$$\text{(c)} \quad \mathrm{P}[\ X > 1\ ] = 1 - \frac{6}{\pi^2} \qquad \text{(d)} \quad \mathrm{P}[\ X > 1\ ] = \frac{e^2 - 3}{e^2 - 1}$$

P[X is even] $= \sum_{x=1}^{\infty}$ P[$X = 2x$], so

$$\text{(a)} \quad \mathrm{P}[\ X \text{ is even}\ ] = \frac{1}{3} \qquad \text{(b)} \quad \mathrm{P}[\ X \text{ is even}\ ] = 1 - \frac{\log 3}{\log 4}$$

$$\text{(c)} \quad \mathrm{P}[\ X \text{ is even}\ ] = \frac{1}{4} \qquad \text{(d)} \quad \mathrm{P}[\ X \text{ is even}\ ] = \frac{1 - e^{-2}}{2}$$

(a) is still the sum of a geometric progression

(b) follows from the logarithmic series expansion;

$$P[X \text{ is even}] = \sum_{x=1}^{\infty} P[X = 2x] = c\sum_{x=1}^{\infty} \frac{1}{(2x)2^{2x}} = \frac{c}{2}\sum_{x=1}^{\infty} \frac{1}{x4^x} = \frac{c}{2} \times (-\log(1 - 1/4))$$

(c) follows from the initial result taking out a factor of 1/4

(d) uses the sum of the two power series of e^t and e^{-t}, to knock out the odd terms, evaluated at $t = 2$.

2. Let Z and X be the numbers of Heads obtained on the first and second tosses respectively. Then the ranges of Z and X are both $\{0, 1, 2, ..., n\}$. Now

$$f_X(x) = \mathrm{P}[\ X = x\ ] = \sum_{z=1}^{n} \mathrm{P}[\ X = x \mid Z = z\ ]\ \mathrm{P}[\ Z = z\ ] = \sum_{z=x}^{n} \binom{z}{x} \left(\frac{1}{2}\right)^z \binom{n}{z} \left(\frac{1}{2}\right)^n$$

using the Theorem of Total probability. Hence

$$f_X(x) = \left(\frac{1}{2}\right)^n \sum_{z=x}^{n} \frac{z!}{x!(z-x)!} \frac{n!}{z!(n-z)!} \left(\frac{1}{2}\right)^z = \left(\frac{1}{2}\right)^n \binom{n}{x} \sum_{z=x}^{n} \binom{n-x}{n-z} \left(\frac{1}{2}\right)^z$$

But

$$\sum_{z=x}^{n} \binom{n-x}{n-z} \left(\frac{1}{2}\right)^z = \sum_{t=0}^{m} \binom{m}{m-t} \left(\frac{1}{2}\right)^{t+x} = \left(\frac{1}{2}\right)^x \left(1 + \frac{1}{2}\right)^m$$

where $t = z - x$, and $m = n - x$, using the Binomial Expansion. Hence

$$f_X(x) = \binom{n}{x} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^x \left(1 + \frac{1}{2}\right)^{n-x} = \binom{n}{x} \frac{3^{n-x}}{2^{2n}} \qquad x = 0, 1, 2, ..., n.$$

Alternately, as all tosses are independent, consider tossing all n coins twice, and counting the number that show heads twice; this is identical to evaluating X. Then as each coin shows heads twice with probability $\left(\frac{1}{2}\right)^2$,

$$f_X(x) = \binom{n}{x} \left\{\left(\frac{1}{2}\right)^2\right\}^x \left\{1 - \left(\frac{1}{2}\right)^2\right\}^{n-x} = \binom{n}{x} \frac{3^{n-x}}{2^{2n}}$$

as before.

3. Each of the $n(n+1)/2$ points has equal probability $p = 2/(n(n+1))$ of being selected. In column x of the triangular array of points, there are x points in total; in row y, there are $(n + 1 - y)$ points (for $x, y = 1, 2, ..., n$) and therefore

$$f_X(x) = \mathrm{P}[\ X = x\ ] = xp = \frac{2x}{n(n+1)} \qquad x = 1, 2, ..., n$$

$$f_Y(y) = \mathrm{P}[\ Y = y\ ] = (n + 1 - y)p = \frac{2(n+1-y)}{n(n+1)} \qquad y = 1, 2, ..., n$$

4. Can calculate F_X by integration

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt = \int_0^x ct^2(1-t)\ dt = c\left[\frac{x^3}{3} - \frac{x^4}{4}\right] \qquad 0 < x < 1$$

and $F_X(1) = 1$ gives $c = 12$. Finally,

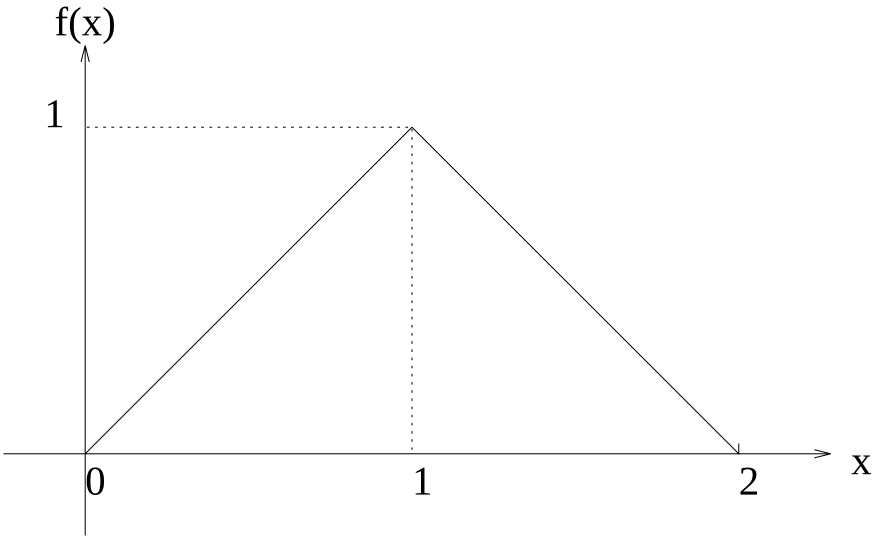
$$\mathrm{P}[\ X > 1/2\ ] = 1 - \mathrm{P}[X \le 1/2] = 1 - F_X(1/2) = 1 - 12[1/24 - 1/64] = 11/16.$$

5. Valid pdf if (i) it is a non-negative function (that is, if $k > 0$), and (ii) integrates to 1 over the range $x > 1$, that is

$$\int_1^\infty f_X(x)\ dx = \int_1^\infty \frac{k}{x^{k+1}}\ dx = \left[-\frac{1}{x^k}\right]_1^\infty = 1 \ \text{ if } k > 0$$

so f_X is a pdf if $k > 0$, and $F_X(x) = 1 - \dfrac{1}{x^k}$ for $x > 1$.

6. Sketch of f_X;



$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt = \begin{cases} \displaystyle\int_0^x t\ dt & = \dfrac{x^2}{2} & 0 < x < 1 \\ \displaystyle\int_0^1 t\ dt + \int_1^x (2-t)\ dt & = 2x - \dfrac{x^2}{2} - 1 & 1 \leq x < 2 \end{cases}$$

Note that F_X is continuous, and $F_X(0) = 0$, $F_X(2) = 1$.

7. Note small error in question: should be

$$F_X(x) = c(\alpha x^\beta - \beta x^\alpha) \qquad 0 \leq x \leq 1$$

and $F_X(x) = 0$ if $x < 0$, **but** $\mathbf{F_X(x) = 1}$ **if** $\mathbf{x > 1}$.

$F_X(1) = 1 \Longrightarrow c = \dfrac{1}{\alpha - \beta}$, and

$$f_X(x) = \frac{d}{dt}\{F_X(t)\}_{t=x} = \frac{\alpha\beta}{\alpha - \beta}\left(x^{\beta-1} - x^{\alpha-1}\right) \qquad 0 \leq x \leq 1$$

and zero otherwise, and hence

$$\begin{aligned} \mathrm{E}_{f_X}[\ X^r\ ] &= \int_{-\infty}^{\infty} x^r f_X(x)\ dx = \int_0^1 \frac{\alpha\beta}{\alpha-\beta} x^r \left(x^{\beta-1} - x^{\alpha-1}\right)\ dx \\ &= \frac{\alpha\beta}{\alpha-\beta}\left[\frac{x^{\beta+r}}{\beta+r} - \frac{x^{\alpha+r}}{\alpha+r}\right]_0^1 \\ &= \frac{\alpha\beta}{(\alpha+r)(\beta+r)} \end{aligned}$$

8. By differentiation,

$$f_X(x) = \frac{d}{dt}\{F_X(t)\}_{t=x} = \frac{2\beta(\beta^2 - x^2)}{(\beta^2 + x^2)^2} \qquad 0 \leq x \leq \beta$$

and zero otherwise, and hence

$$\begin{aligned}
\mathrm{E}_{f_X}[\,X\,] &= \int_{-\infty}^{\infty} x f_X(x)\;dx = \int_0^{\beta} x \frac{2\beta(\beta^2 - x^2)}{(\beta^2 + x^2)^2}\;dx \\
&= \int_0^{\pi/4} 2\beta^2 \tan\theta \frac{\beta^2(1 - \tan^2\theta)}{\beta^4(1 + \tan^2\theta)^2} \beta \sec^2\theta\;d\theta \qquad (x = \beta\tan\theta) \\
&= 2\beta \int_0^{\pi/4} \tan\theta\;\frac{(1 - \tan^2\theta)}{(1 + \tan^2\theta)}\;d\theta \\
&= 2\beta \int_0^{\pi/4} \tan\theta \cos 2\theta\;d\theta \\
&= 2\beta \left[\frac{1}{2}\tan\theta \sin 2\theta\right]_0^{\pi/4} - \beta \int_0^{\pi/4} \sec^2\theta \sin 2\theta\;d\theta \qquad \text{(by parts)} \\
&= 2\beta \left[\frac{1}{2} - \int_0^{\pi/4} \tan\theta\;d\theta\right] \\
&= 2\beta \left[\frac{1}{2} - [-\log(\cos\theta)]_0^{\pi/4}\right] \\
&= 2\beta \left[\frac{1}{2} + \log(\cos\pi/4)\right] = \beta(1 - \log 2)
\end{aligned}$$

as $\cos\pi/4 = 1/\sqrt{2}$.