M2S1 - EXERCISES 1: SOLUTIONS

1. This equation can only hold if all terms are well-defined. Thus we must have $P(B) > 0$ and $P(B') > 0$ in order for the conditional probabilities to exist. Hence we know that $\emptyset \subset B \subset \Omega$. Furthermore, we have by the Theorem of Total Probability that

$$P(A) = P(A \cap B) + P(A \cap B') = P(A|B)\,P(B) + P(A|B')\,P(B').$$

Hence

$$P(A) - \{P(A|B) + P(A|B')\} = P(A|B)\{P(B) - 1\} + P(A|B')\{P(B') - 1\}$$

and as we have established $0 < P(B), P(B') < 1$, the RHS is no greater than zero. Hence the equality only holds if

$$P(A) = P(A|B) = P(A|B') = 0$$

that is, if $P(A) = 0$.

2. Let H_n be the event that n tosses result in an even number of heads. Conditioning on the result after $n-1$ tosses, and using the Theorem of Total Probability

$$P(H_n) = P(H_n|H_{n-1})P(H_{n-1}) + P(H_n|H_{n-1}^{'})P(H_{n-1}^{'})$$

Therefore,

$$p_n = (1-p)p_{n-1} + p(1-p_{n-1}) = p + (1-2p)p_{n-1}$$

Now, to find a solution to this difference equation, propose $p_n = A + B\lambda^n$ for all $n \geq 0$. Then

$$\left.\begin{array}{ll} n = 0 & p_0 = A + B = 1 \\ \\ n \geq 1 & p_n = A + B\lambda^n = p + (1-2p)(A + B\lambda^{n-1}) \end{array}\right\} \Longrightarrow \lambda = (1-2p), A = B = \frac{1}{2},\ p_n = \frac{1}{2} + \frac{1}{2}(1-2p)^n.$$

If $p < 1/2$, $(1-2p) > 0$, so $p_n > 1/2$ for all n.

As $n \longrightarrow \infty$,

$$p_n \longrightarrow \left\{\begin{array}{ll} 1/2 & 0 < p < 1 \\ 1 & p = 0 \end{array}\right.$$

and if $p = 1$ no limit exists.

3. Let F_n be the event that the weather is fine on day n. Then conditioning on the weather on day $n-1$, and using the Theorem of Total Probability

$$P(F_n) = P(F_n|F_{n-1})P(F_{n-1}) + P(F_n|F_{n-1}^{'})P(F_{n-1}^{'}) \qquad \theta_n = p\theta_{n-1} + (1-p)(1-\theta_{n-1})$$

and hence

$$\left(\theta_n - \frac{1}{2}\right) = (2p-1)\left(\theta_{n-1} - \frac{1}{2}\right) = (2p-1)^{n-1}\left(\theta_1 - \frac{1}{2}\right) \qquad \theta_n = \frac{1}{2} + (2p-1)^{n-1}\left(\theta - \frac{1}{2}\right)$$

so $\theta_n \longrightarrow \frac{1}{2}$ as $n \longrightarrow \infty$.

4. Let E and F be the events that the sequence of tosses results in n Heads, and that the coin is fair respectively. Then

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^{'})P(F^{'})}$$

(i) $P(E|F) = \left(\frac{1}{2}\right)^n$, $P(E|F^{'}) = 1$, $P(F) = P(F^{'}) = \frac{1}{2}$, and hence $P(F|E) = \dfrac{1}{1+2^n}$.

(ii) $P(E|F) = \binom{n}{k}\left(\frac{1}{2}\right)^n$, $P(E|F^{'}) = \binom{n}{k}p^k(1-p)^{n-k}$, $P(F) = P(F^{'}) = \frac{1}{2}$, and hence

$$P(F|E) = \frac{1}{1 + 2^n p^k (1-p)^{n-k}}.$$

5. Let E, F and G be the events that the flower produces ripe fruit, that the flower is pollinated, and that the fruit ripens respectively. Then $P(E) = P(F \cap G) = P(F)P(G) = \frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$.

Now let A_n be the event that the tree produces n flowers, and B_r be the event that the tree produces r ripe fruit (for $n \geq r$). Then

$$P(A_n|B_r) = \frac{P(B_r|A_n)P(A_n)}{\sum\limits_{n=r}^{\infty} P(B_r|A_n)P(A_n)}$$

Now

$$P(B_r|A_n) = \binom{n}{r}\left(\frac{1}{2}\right)^n \qquad P(A_n) = (1-p)p^n$$

so

$$\begin{aligned} P(A_n|B_r) &= \frac{\binom{n}{r}\left(\frac{1}{2}\right)^n (1-p)p^n}{\sum\limits_{n=r}^{\infty}\binom{n}{r}\left(\frac{1}{2}\right)^n (1-p)p^n} \\ &= \binom{n}{r}\left(\frac{p}{2}\right)^n \frac{1}{\sum\limits_{x=0}^{\infty}\binom{x+r}{r}\left(\frac{p}{2}\right)^{x+r}} = \binom{n}{r}\left(\frac{p}{2}\right)^{n-r} \frac{1}{\left(1-\frac{p}{2}\right)^{-(r+1)}} \end{aligned}$$

using the binomial expansion for negative exponent. Hence

$$P(A_n|B_r) = \binom{n}{r}\frac{p^{n-r}(2-p)^{r+1}}{2^{n+1}} \qquad r \leq n$$

6. Let T_k be the event that there are k successive positive tests, let S be the event that drugs are present. Then

$$P(S|T_k) = \frac{P(T_k|S)P(S)}{P(T_k|S)P(S) + P(T_k|S')P(S')} = \frac{0.99^k \times 0.0002}{0.99^k \times 0.0002 + (1-0.98)^k \times (1-0.0002)}$$

as, by conditional independence

$$P(T_k|S) = \{P(T_1|S)\}^k \qquad P(T_k|S') = \{P(T_1|S')\}^k$$

If $k = 1$, $P(S|T_1) = 0.0098$.

If $k = 2$, $P(S|T_2) = 0.3289$.