M2S1 - ASSESSED COURSEWORK 2 SOLUTIONS

(a) For the random variables given:

(i) Given $R = r$ with $0 < r < 1$, we require that,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y|R}(x,y|r)dx\,dy = 1 \qquad \therefore \qquad \int_{y=-r}^{y=r}\left\{\int_{x=-\sqrt{r^2-y^2}}^{x=\sqrt{r^2-y^2}} k(r)x^2y^2\,dx\right\}dy = 1.$$

Now

$$\begin{aligned}
\int_{-r}^{r}\left\{\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} k(r)x^2y^2\,dx\right\}dy &= k(r)\int_{-r}^{r} y^2\left\{\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} x^2\,dx\right\}dy \\
&= k(r)\int_{-r}^{r} y^2\left[\frac{1}{3}x^3\right]_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dy \\
&= \frac{2k(r)}{3}\int_{-r}^{r} y^2(r^2-y^2)^{3/2}\,dy
\end{aligned}$$

Making the substitution $y = r\cos t$ in this integral, we obtain

$$\begin{aligned}
\int_{-r}^{r} y^2(r^2-y^2)^{3/2}\,dy &= \int_0^{\pi} r^2\cos^2 t\,(r^2 - r^2\cos^2 t)^{3/2} r\sin t\,dt = r^6\int_0^{\pi}\cos^2 t\,\sin^4 t\,dt \\
&= r^6\left[\cos t\,\frac{\sin^5 t}{5}\right]_0^{\pi} + \frac{1}{5}r^6\int_0^{\pi}\sin^6 t\,dt. \\
&= 0 + \frac{1}{5}r^6 I(6),
\end{aligned}$$

say, where

$$I(k) = \int_0^{\pi}\sin^k t\,dt.$$

Now, integrating by parts, we obtain a recursion for $I(k)$;

$$\begin{aligned}
I(k) = \int_0^{\pi}\sin^k t\,dt = \int_0^{\pi}\sin^2 t\,\sin^{k-2} t\,dt &= \int_0^{\pi}(1-\cos^2 t)\,\sin^{k-2} t\,dt \\
&= \int_0^{\pi}\sin^{k-2} t\,dt - \int_0^{\pi}\cos^2 t\,\sin^{k-2} t\,dt \\
&= I(k-2) - \left[\cos t\frac{\sin^{k-1} t}{k-1}\right]_0^{\pi} - \frac{1}{k-1}\int_0^{\pi}\sin^k t\,dt \\
&= I(k-2) - 0 - \frac{1}{k-1}I(k)
\end{aligned}$$

so that

$$I(k) = \left(\frac{k-1}{k}\right)I(k-2) \tag{1}$$

and hence

$$I(6) = \left(\frac{5}{6}\right)I(4) = \left(\frac{5}{6}\right)\left(\frac{3}{4}\right)I(2) = \left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)I(0) = \frac{15}{48}I(0).$$

But $I(0) = \pi$ by direct evaluation, so therefore $I(6) = 15\pi/48 = 5\pi/16$. Hence, from above

$$\int_{-r}^{r} y^2(r^2 - y^2)^{3/2}\, dy = \frac{1}{5} r^6 I(6) = \frac{\pi r^6}{16}$$

and thus, for $0 < r < 1$,

$$k(r) = \frac{3}{2}\frac{16}{\pi r^6} = \frac{24}{\pi r^6}.$$

Derivation of the recursion formula in equation (1) is not necessary for full marks, but can be quoted as a standard result. Using a cos substitution is also OK, you just get a similar recursion, and may give the answer more quickly. Please give some marks for correctly formulating the problem, even if the script does not complete the integration.

[6 MARKS]

(ii) The full joint pdf is therefore

$$f_{R,X,Y}(r,x,y) = f_{X,Y|R}(x,y|r) f_R(r) = \frac{24x^2y^2}{\pi r^6}\, 4r^3 = \frac{96x^2y^2}{\pi r^3}$$

on the region defined by

$$-r < x < r,\ -r < y < r,\ 0 < x^2 + y^2 < r^2,\ 0 < r < 1,$$

and zero otherwise. To get the joint marginal for X and Y, we integrate out R from the full joint pdf, that is

$$\begin{aligned} f_{X,Y}(x,y) = \int_{-\infty}^{\infty} f_{R,X,Y}(r,x,y)\, dr = \int_{\sqrt{x^2+y^2}}^{1} \frac{96x^2y^2}{\pi r^3}\, dr &= \frac{96x^2y^2}{\pi}\left[-\frac{1}{2r^2}\right]_{\sqrt{x^2+y^2}}^{1} \\ &= \frac{48x^2y^2}{\pi}\left[\frac{1}{x^2+y^2} - 1\right] \\ &= \frac{48x^2y^2(1 - x^2 - y^2)}{\pi(x^2+y^2)}. \end{aligned}$$

on the region defined by

$$0 < x < 1,\ 0 < y < 1,\ 0 < x^2 + y^2 < 1$$

and zero otherwise. *[4 MARKS]*

(b) Using the joint pdf given:

(i) From first principles, for fixed $y_1 > 0$

$$F_{Y_1}(y_1) = P[Y_1 \le y_1] = P[X_1/X_2 \le y_1] = P[X_1 \le y_1 X_2].$$

Hence

$$\begin{aligned} F_{Y_1}(y_1) &= \int_0^\infty \int_{x_1/y_1}^\infty \lambda_1\lambda_2 \exp\{-(\lambda_1 x_1 + \lambda_2 x_2)\}\, dx_2\, dx_1 \\ &= \int_0^\infty \lambda_1 e^{-\lambda_1 x_1} \left\{\int_{x_1/y_1}^\infty \lambda_2 e^{-\lambda_2 x_2}\, dx_2\right\} dx_1 \\ &= \int_0^\infty \lambda_1 e^{-\lambda_1 x_1}\, e^{-\lambda_2 x_1/y_1} dx_1 = \int_0^\infty \lambda_1 e^{-(\lambda_1 + \lambda_2/y_1)x_1}\, dx_1 \\ &= \frac{\lambda_1}{(\lambda_1 + \lambda_2/y_1)} \end{aligned}$$

[3 MARKS]

(ii)

$$P[Y_1 < Y_2] = P[X_1/X_2 < X_1X_2] = P[X_2^2 > 1] = P[X_2 > 1]$$

as X_2 is strictly positive. Hence

$$P[Y_1 < Y_2] = 1 - F_{X_2}(1) = \exp\{-\lambda_2\}.$$

[2 MARKS]

(iii) For the covariance, from first principles

$$Cov_{f_{X_2,Y_2}}[X_2, Y_2] = E_{f_{X_2,Y_2}}[X_2\ Y_2] - E_{f_{X_2}}[X_2]E_{f_{Y_2}}[Y_2]$$

By direct calculation

$$E_{f_{X_2}}[X_2] = 1/\lambda_2$$

and by definition

$$E_{f_{Y_2}}[Y_2] \equiv \int_0^\infty \int_0^\infty x_1x_2 f_{X_1,X_2}(x_1,x_2)\ dx_1\ dx_2 \equiv E_{f_{X_1}}[X_1]E_{f_{X_2}}[X_2] = \frac{1}{\lambda_1\lambda_2}.$$

Finally,

$$\begin{aligned} E_{f_{X_2,Y_2}}[X_2\ Y_2] &= \int_0^\infty \int_0^\infty x_2y_2 f_{X_2,Y_2}(x_2,y_2)\ dx_2\ dy_2 \\ &\equiv \int_0^\infty \int_0^\infty x_1x_2^2 f_{X_1,X_2}(x_1,x_2)\ dx_1\ dx_2 \\ &= E_{f_{X_1}}[X_1]E_{f_{X_2}}[X_2^2] = \frac{1}{\lambda_1}\frac{2}{\lambda_2^2}. \end{aligned}$$

Hence

$$Cov_{f_{X_2,Y_2}}[X_2, Y_2] = \frac{2}{\lambda_1\lambda_2^2} - \frac{1}{\lambda_2}\frac{1}{\lambda_1\lambda_2} = \frac{1}{\lambda_1\lambda_2^2}$$

[5 MARKS]

Note: for the r^{th} moment for X_1 or X_2; proceed as follows

$$\begin{aligned} E_{f_X}[X^r] = \int_0^\infty x^r\lambda e^{-\lambda x}\ dx = \left[-x^r e^{-\lambda x}\right]_0^\infty + \int_0^\infty r x^{r-1} e^{-\lambda x}\ dx &= r\int_0^\infty x^{r-1}e^{-\lambda x}\ dx \\ &= \frac{r}{\lambda}E_{f_X}[X^{r-1}] \\ &= \frac{r(r-1)}{\lambda^2}E_{f_X}[X^{r-2}] \\ &\ \ \vdots \\ &= \frac{r!}{\lambda^r}E_{f_X}[X^0] \\ &= \frac{r!}{\lambda^r}. \end{aligned}$$