M2S1 - ASSESSED COURSEWORK 1 SOLUTIONS

(a) We have

$$f_N(n) = k_1 \frac{(-\log(1-\phi))^n}{n!} \qquad n = 0, 1, 2, \ldots$$

and zero otherwise, for some parameter ϕ, and constant k_1.

(i) This sum is convergent and non-negative if and only if $0 < \phi < 1$; this follows by inspection of f_N.

[1 MARK]

(ii) Writing $\lambda = -\log(1-\phi)$, and noting that

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = \exp\{\lambda\}$$

by the exponential series sum formula, so

$$\sum_{n=0}^{\infty} f_N(n) = 1 \qquad \Longrightarrow \qquad k_1 = \exp\{-\lambda\} = \exp\{-(-\log(1-\phi))\} = 1 - \phi.$$

[2 MARKS]

(iii)

$$P[N > 0] = 1 - P[N = 0] = 1 - (1-\phi)\frac{(-\log(1-\phi))^0}{0!} = 1 - (1-\phi) = \phi.$$

[2 MARKS]

(b) We now have a continuous variable, with pdf f_X given by

$$f_X(x) = k_2 x \exp\{-\beta x\} \qquad x > 0$$

and zero otherwise, for parameter $\beta > 0$, and constant k_2.

(i)

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \qquad \Longrightarrow \qquad k_2 = \left[\int_0^{\infty} x\exp\{-\beta x\}dx\right]^{-1}.$$

By parts

$$\begin{aligned}
\int_0^{\infty} x\exp\{-\beta x\}dx &= \left[-\frac{x}{\beta}\exp\{-\beta x\}\right]_0^{\infty} + \frac{1}{\beta}\int_0^{\infty}\exp\{-\beta x\}dx \\
&= 0 + \frac{1}{\beta}\left[-\frac{1}{\beta}\exp\{-\beta x\}\right]_0^{\infty} \\
&= \frac{1}{\beta^2}
\end{aligned}$$

so $k_2 = \beta^2$.

[1 MARK]

(ii) Using similar techniques, for $x > 0$,

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)dt = \int_0^x \beta^2 x \exp\{-\beta x\}dx = [-\beta t \exp\{-\beta t\}]_0^x + \beta \int_0^x \exp\{-\beta t\}dt \\ &= -\beta x \exp\{-\beta x\} + \beta \left[-\frac{1}{\beta}\exp\{-\beta x\}\right]_0^x \\ &= -\beta x \exp\{-\beta x\} + 1 - \exp\{-\beta x\} = 1 - (1+\beta x)\exp\{-\beta x\} \end{aligned}$$

[2 MARKS]

(iii) $P[X > x] = 1 - P[X \leq x] = 1 - F_X(x) = (1+\beta x)\exp\{-\beta x\}$.

[2 MARKS]

(c) In this problem, T is a positive random variable, with pdf f_T say. Using the partition given and the Theorem of Total Probability, taking probabilities on both sides, we have the cdf of T defined by

$$F_T(t) = P[T \leq t] = \sum_{n=0}^{\infty} P[T \leq t | N = n] P[N = n]$$

and on differentiation we get the pdf

$$f_T(t) = \sum_{n=0}^{\infty} f_T(t|N = n) f_N(n)$$

where $f_T(t|N = n)$ is the pdf of T **if we know that** $N = n$[1]. Thus, for the mgf of T, we have (using s as the argument instead of t to avoid confusion)

$$\begin{aligned} M_T(s) = E_{f_T}[e^{sT}] = \int_0^\infty e^{sT} f_T(t)dt &= \int_0^\infty e^{sT} \left\{ \sum_{n=0}^{\infty} f_T(t|N = n) f_N(n) \right\} dt \\ &= \sum_{n=0}^{\infty} \left\{ \int_{t=0}^{\infty} e^{sT} f_T(t|N = n)dt \right\} f_N(n) \\ &= \sum_{n=0}^{\infty} E_{f_T}[e^{sT}|N = n] f_N(n) = \sum_{n=0}^{\infty} M_T(s|N = n) f_N(n), \end{aligned}$$

say, where $M_T(s|N = n) = E_{f_T}[e^{sT}|N = n]$ is the mgf for T **if we know that** $N = n$.

[3 MARKS]

Now, using the key mgf result, and the hint given, we know that **if** $N = n$, then for argument s in a suitable neighbourhood.

$$T = \sum_{i=1}^{n} X_i \qquad \Longrightarrow \qquad M_T(s|N = n) = \{M_X(s)\}^n$$

where M_X is the mgf of $X_1, \ldots, X_n$, as these variables are independent and identically distributed.

[1 MARK]

[1] In reality, of course, we do not know N, as it is a random quantity; this is why we have to use the partition.

Thus, from above, again using $\lambda = -\log(1-\phi)$, we have the mgf of T as

$$\begin{aligned} M_T(s) = \sum_{n=0}^{\infty} M_T(s|N=n) f_N(n) = \sum_{n=0}^{\infty} \{M_X(s)\}^n \exp\{-\lambda\} \frac{\lambda^n}{n!} &= \exp\{-\lambda\} \sum_{n=0}^{\infty} \frac{(\lambda M_X(s))^n}{n!} \\ &= \exp\{-\lambda\} \exp\{(\lambda M_X(s))\} \\ &= \exp\{\lambda (M_X(s) - 1)\}. \end{aligned}$$

[2 MARKS]

Now, by direct calculation

$$M_X(s) = \int_0^{\infty} e^{sx} f_X(x) dx = \int_0^{\infty} e^{sx} \beta^2 x \exp\{-\beta x\} dx = \beta^2 \int_0^{\infty} x \exp\{-x(\beta - s)\} dx = \left(\frac{\beta}{\beta - s}\right)^2$$

for $\beta > s$, as the integrand is proportional to the pdf f_X. Thus, differentiating twice yields

$$M_X^{(1)}(s) = 2\beta^2(\beta - s)^{-3} \quad \therefore \quad M_X^{(1)}(0) = \frac{2}{\beta}$$

$$M_X^{(2)}(s) = 6\beta^2(\beta - s)^{-4} \quad \therefore \quad M_X^{(2)}(0) = \frac{6}{\beta^2}$$

and finally, for T, from above

$$M_T^{(1)}(s) = \lambda M_X^{(1)}(s) \exp\{\lambda(M_X(s) - 1)\} \ \therefore \ M_T^{(1)}(0) = \lambda M_X^{(1)}(0) \exp\{\lambda(M_X(0) - 1)\} = \lambda \frac{2}{\beta}$$

as $M_X(0) = 1$. Similarly

$$M_T^{(2)}(s) = \lambda^2 (M_X^{(1)}(s))^2 \exp\{\lambda(M_X(s) - 1)\} + \lambda M_X^{(2)}(s) \exp\{\lambda(M_X(s) - 1)\}$$

so that

$$M_T^{(2)}(0) \quad = \quad \lambda^2 (M_X^{(1)}(0))^2 \exp\{\lambda(M_X(0) - 1)\} + \lambda M_X^{(2)}(0) \exp\{\lambda(M_X(0) - 1)\} = \lambda^2 \frac{4}{\beta^2} + \lambda \frac{6}{\beta^2}$$

and hence

$$E_{f_T}[T] = \frac{2\lambda}{\beta} \qquad E_{f_T}[T^2] = \frac{4\lambda^2}{\beta^2} + \frac{6\lambda}{\beta^2}$$

and hence

$$Var_{f_T}[T] = E_{f_T}[T^2] - \{E_{f_T}[T]\}^2 = \frac{4\lambda^2}{\beta^2} + \frac{6\lambda}{\beta^2} - \frac{4\lambda^2}{\beta^2} = \frac{6\lambda}{\beta^2}.$$

[4 MARKS]