

WORKED EXAMPLES 5

CONVERGENCE IN DISTRIBUTION

EXAMPLE 1: Continuous random variable X with range $\mathbb{X} \equiv (0, n]$ for $n > 0$ and cdf

$$F_{X_n}(x) = 1 - \left(1 - \frac{x}{n}\right)^n \quad 0 < x \leq n$$

and zero otherwise. Then as $n \rightarrow \infty$, $\mathbb{X} \equiv (0, \infty)$, and for all $x > 0$

$$F_{X_n}(x) \rightarrow 1 - e^{-x} \quad \therefore \quad F_{X_n}(x) \rightarrow F_X(x) = 1 - e^{-x}$$

and hence

$$X_n \xrightarrow{d} X \quad X \sim \text{Exponential}(1)$$

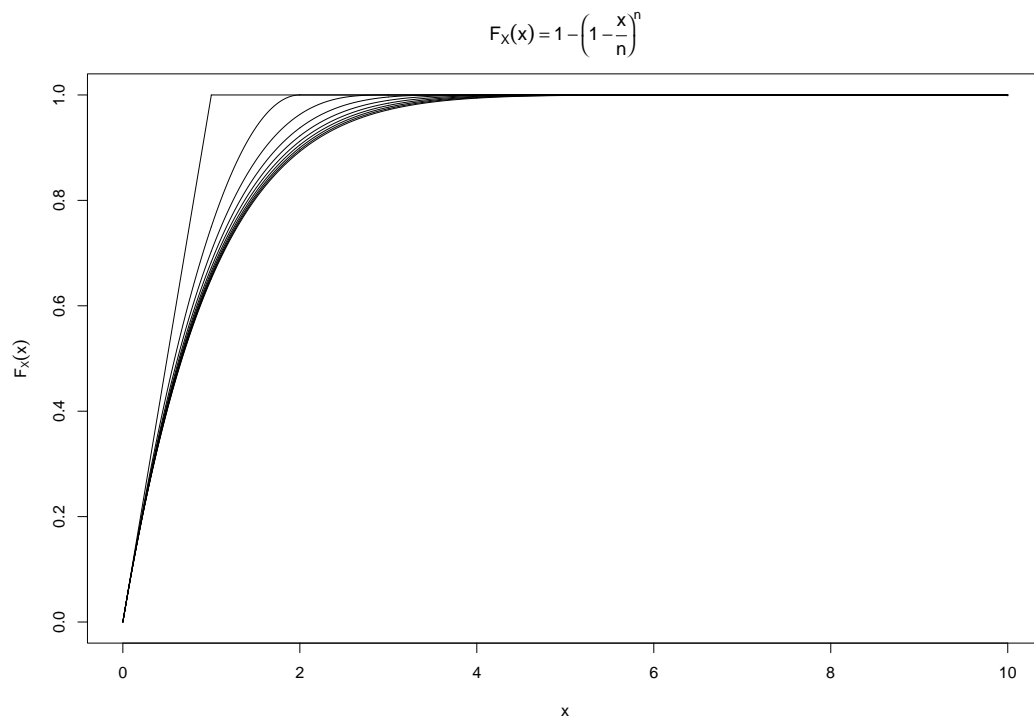


Figure 1: Example 1: for $n = 1, \dots, 10$.

EXAMPLE 2: Continuous random variable X with range $\mathbb{X} \equiv (0, \infty)$ and cdf

$$F_{X_n}(x) = \left(1 - \frac{1}{1+nx}\right)^n \quad 0 < x < \infty$$

and zero otherwise. Then as $n \rightarrow \infty$, for all $x > 0$

$$F_{X_n}(x) \rightarrow e^{-1/x} \quad \therefore \quad F_{X_n}(x) \rightarrow F_X(x) = e^{-1/x}$$

as

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{1+nx}\right)^n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{nx}\right)^n = \lim_{n \rightarrow \infty} \left(1 - \frac{1/x}{n}\right)^n$$

and for any z

$$\lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

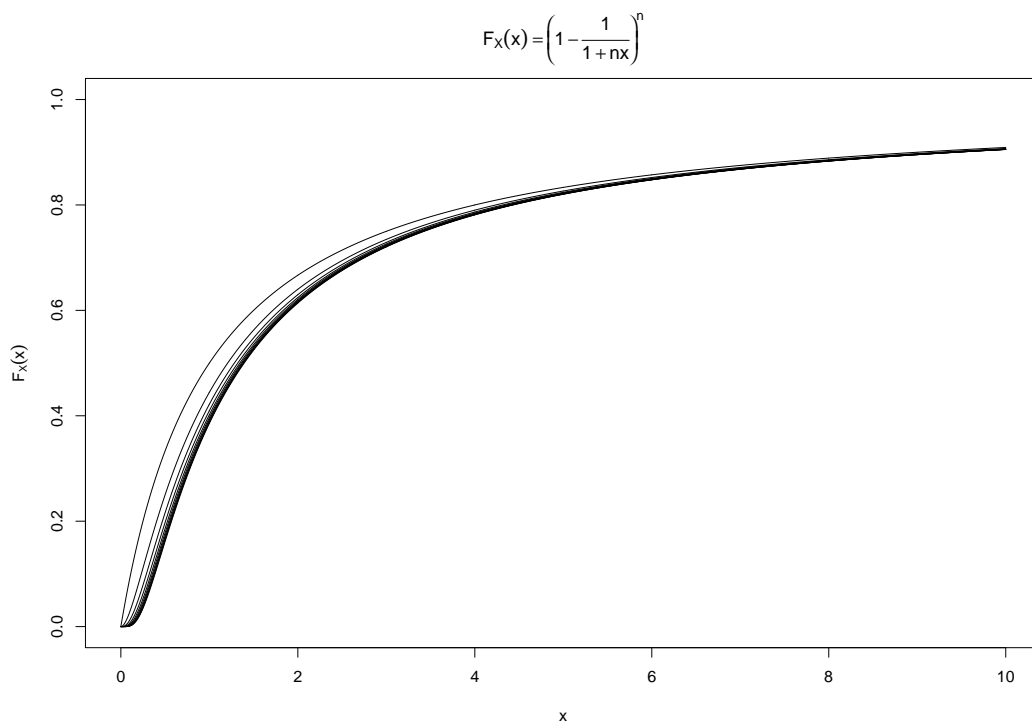


Figure 2: Example 2: for $n = 1, \dots, 10$.

EXAMPLE 3: Continuous random variable X with range $\mathbb{X} \equiv [0, 1]$ and cdf

$$F_{X_n}(x) = x - \frac{\sin(2n\pi x)}{2n\pi} \quad 0 \leq x \leq 1$$

and zero otherwise. Then as $n \rightarrow \infty$, and for all $0 \leq x \leq 1$

$$F_{X_n}(x) \rightarrow x \quad \therefore \quad F_{X_n}(x) \rightarrow F_X(x) = x$$

and hence

$$X_n \xrightarrow{d} X \quad \text{where } X \sim \text{Uniform}(0, 1)$$

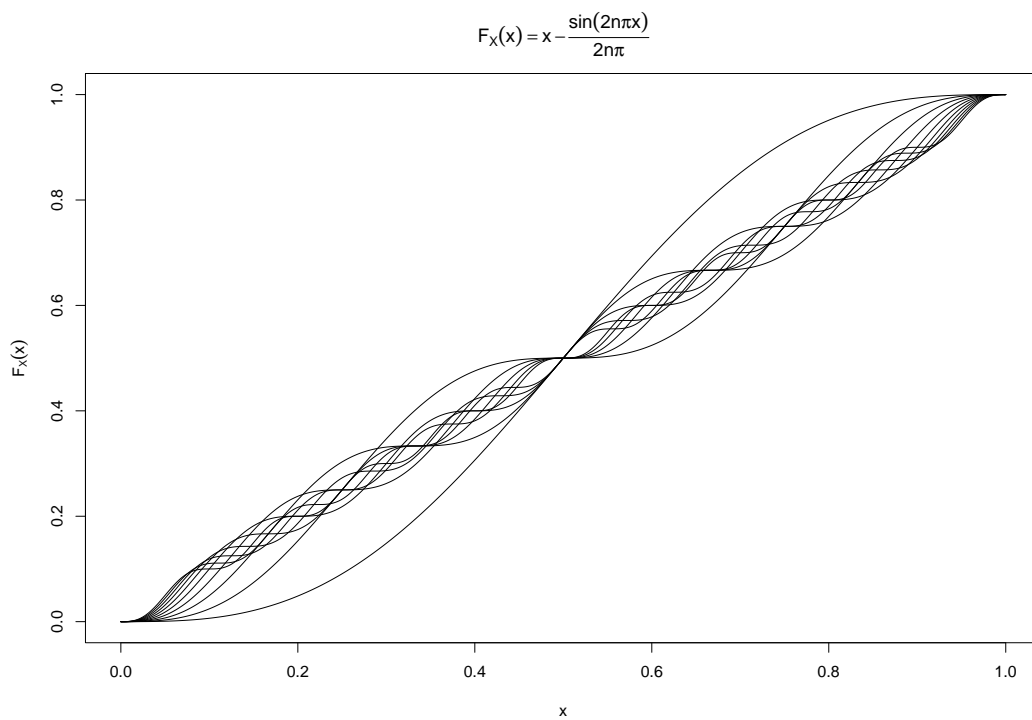


Figure 3: Example 1: for $n = 1, \dots, 10$.

NOTE: for the pdf

$$f_{X_n}(x) = 1 - \cos(2n\pi x) \quad 0 \leq x \leq 1$$

and there **is no limit** as $n \rightarrow \infty$.

EXAMPLE 4: Continuous random variable X with range $\mathbb{X} \equiv [0, 1]$ and cdf

$$F_{X_n}(x) = 1 - (1 - x)^n \quad 0 \leq x \leq 1$$

and zero otherwise. Then as $n \rightarrow \infty$, and for $x \in \mathbb{R}$

$$F_{X_n}(x) \rightarrow \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases} .$$

This limiting form is **not** continuous at $x = 0$, as $x = 0$ is not a point of continuity, and the **ordinary definition of convergence in distribution cannot be applied**. However, it is clear that for $\epsilon > 0$,

$$P[|X| < \epsilon] = 1 - (1 - \epsilon)^n \rightarrow 1 \text{ as } n \rightarrow \infty$$

so it is still correct to say

$$X_n \xrightarrow{d} X \quad \text{where } P[X = 0] = 1$$

so the limiting distribution is **degenerate at** $x = 0$.

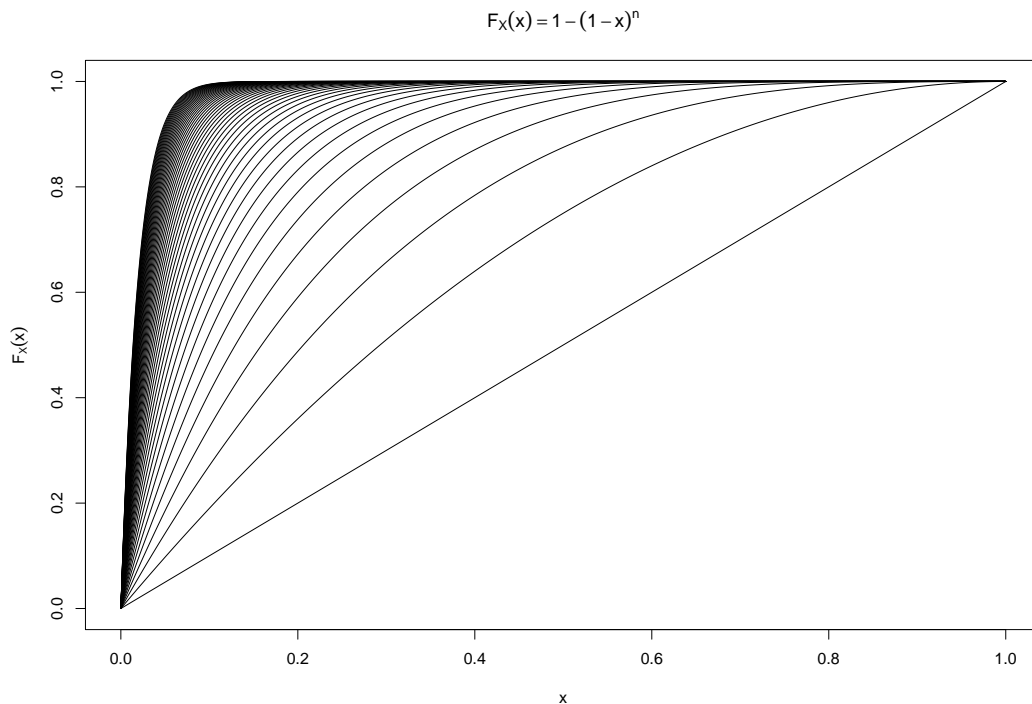


Figure 4: Example 1: for $n = 1, \dots, 50$.

EXAMPLE 5: Continuous random variable X with range $\mathbb{X} \equiv (0, \infty)$ and cdf

$$F_{X_n}(x) = \left(\frac{x}{1+x} \right)^n \quad x > 0$$

and zero otherwise. Then as $n \rightarrow \infty$, and for $x > 0$

$$F_{X_n}(x) \rightarrow 0$$

Thus there is **no limiting distribution**.

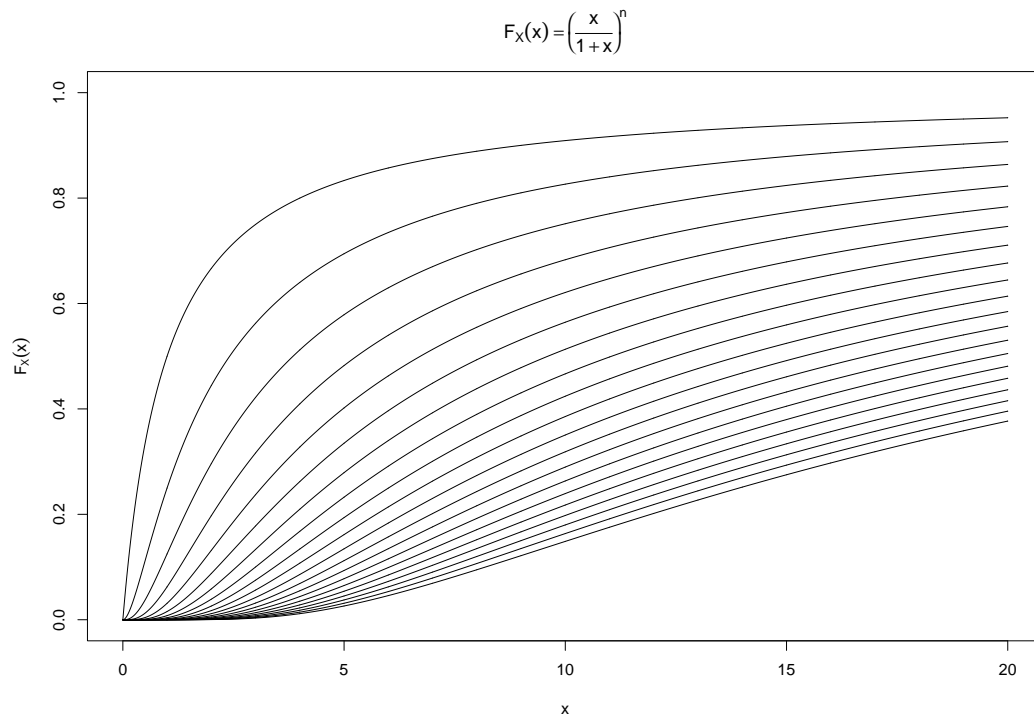


Figure 5: Example 1: for $n = 1, \dots, 20$.

Now let $V_n = X_n/n$. Then $\mathbb{V} \equiv (0, \infty)$ and the cdf of V_n is

$$F_{V_n}(v) = P[V_n \leq v] = P[X_n/n \leq v] = P[X_n \leq nv] = F_{X_n}(nv) = \left(\frac{nv}{1+nv} \right)^n \quad v > 0$$

and as $n \rightarrow \infty$, for all $v > 0$

$$F_{V_n}(v) \rightarrow e^{-1/v} \quad \therefore \quad F_{V_n}(v) \rightarrow F_V(v) = e^{-1/v}$$

and the limiting distribution of V_n does exist.

EXAMPLE 6: Continuous random variable X with range $\mathbb{X} \equiv (0, \infty)$ and cdf

$$F_{X_n}(x) = \frac{\exp(nx)}{1 + \exp(nx)} \quad x \in \mathbb{R}$$

and zero otherwise. Then as $n \rightarrow \infty$

$$F_{X_n}(x) \rightarrow \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} \quad x \in \mathbb{R}$$

This limiting form is **not** a cdf, as it is not right continuous at $x = 0$. However, as $x = 0$ is not a point of continuity, and the ordinary definition of convergence in distribution does not apply. However, it is clear that for $\epsilon > 0$,

$$P[|X| < \epsilon] = \frac{\exp(n\epsilon)}{1 + \exp(n\epsilon)} - \frac{\exp(-n\epsilon)}{1 + \exp(-n\epsilon)} \rightarrow 1 \text{ as } n \rightarrow \infty$$

so it is still correct to say

$$X_n \xrightarrow{d} X \quad \text{where } P[X = 0] = 1$$

and the limiting distribution is **degenerate at** $x = 0$.

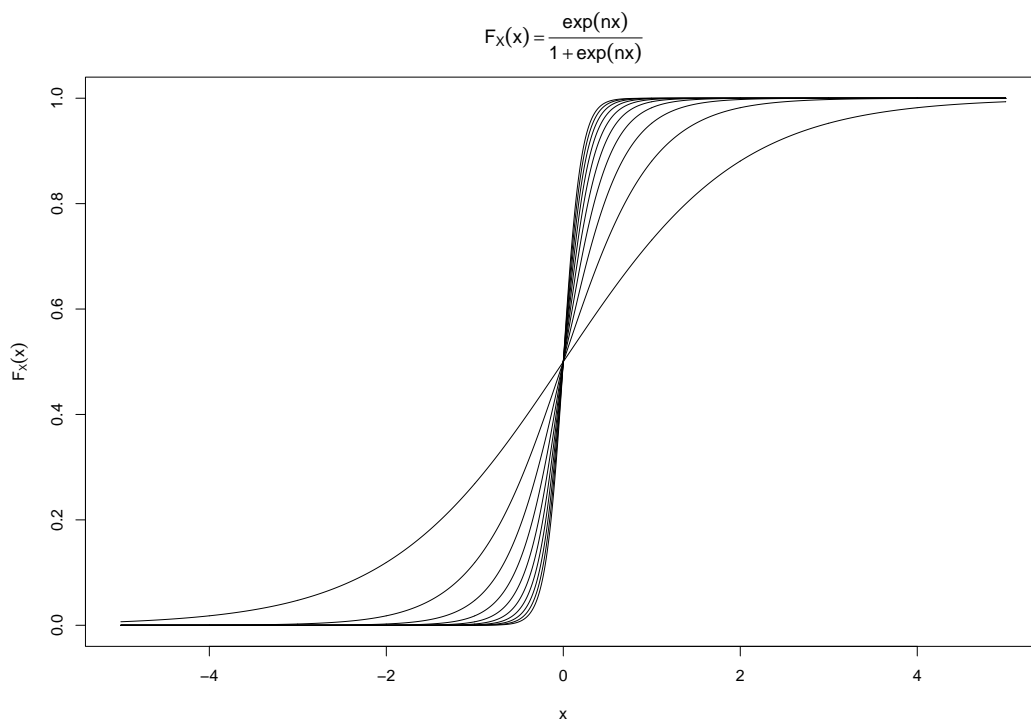


Figure 6: Example 1: for $n = 1, \dots, 10$.