WORKED EXAMPLES 2

CALCULATIONS FOR MULTIVARIATE DISTRIBUTIONS

EXAMPLE 1 Let X_1 and X_2 be discrete random variables each with range $\{1, 2, 3, ...\}$ and joint mass function

$$f_{X_1,X_2}(x_1, x_2) = \frac{c}{(x_1 + x_2 - 1)(x_1 + x_2)(x_1 + x_2 + 1)} \qquad x_1, x_2 = 1, 2, 3, ...$$

and zero otherwise. The marginal mass function for X is given by

$$\begin{aligned} f_{X_1}(x_1) = \sum_{x_2=-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2) &= \sum_{x_2=1}^{\infty} \frac{c}{(x_1 + x_2 - 1)(x_1 + x_2)(x_1 + x_2 + 1)} \\ &= \sum_{x_2=1}^{\infty} \frac{c}{2}\left[\frac{1}{(x_1 + x_2 - 1)(x_1 + x_2)} - \frac{1}{(x_1 + x_2)(x_1 + x_2 + 1)}\right] \\ &= \frac{c}{2}\frac{1}{x_1(x_1 + 1)} \end{aligned}$$

as all other terms cancel, and to calculate c, note that

$$\sum_{x_1=-\infty}^{\infty} f_{X_1}(x_1) = \sum_{x_1=1}^{\infty} \frac{c}{2}\frac{1}{x_1(x_1 + 1)} = \frac{c}{2}\sum_{x_1=1}^{\infty} \left[\frac{1}{x_1} - \frac{1}{x_1 + 1}\right] = \frac{c}{2}$$

as all terms in the sum except the first cancel. Hence $c = 2$. Also, as the joint function is symmetric in form for X_1 and X_2, f_{X_1} and f_{X_2} are identical.

EXAMPLE 2 Let X_1 and X_2 be continuous random variables with ranges $\mathbb{X}_1 = \mathbb{X}_2 = (0, 1)$ and joint pdf defined by

$$f_{X_1,X_2}(x_1, x_2) = 4x_1x_2 \qquad 0 < x_1 < 1,\ 0 < x_2 < 1$$

and zero otherwise. For $0 < x_1, x_2 < 1$,

$$\begin{aligned} F_{X_1,X_2}(x_1, x_2) &= \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} f_{X_1,X_2}(t_1, t_2)\ dt_1dt_2 = \int_{0}^{x_2}\int_{0}^{x_1} 4t_1t_2\ dt_1dt_2 \\ &= \left\{\int_0^{x_1} 2t_1\ dt_1\right\}\left\{\int_0^{x_2} 2t_2\ dt_2\right\} = (x_1x_2)^2 \end{aligned}$$

and a full specification for F_{X_1,X_2} is

$$F_{X_1,X_2}(x_1, x_2) = \begin{cases} 0 & x_1, x_2 \le 0 \\ (x_1x_2)^2 & 0 < x_1, x_2 < 1 \\ x_1^2 & 0 < x_1 < 1, x_2 \ge 1 \\ x_2^2 & 0 < x_2 < 1, x_1 \ge 1 \\ 1 & x_1, x_2 \ge 1 \end{cases}$$

To calculate

$$P\left[\frac{X_1 + X_2}{2} < c\right]$$

we need to integrate f_{X_1,X_2} over the set $A_c = \{(x_1, x_2) : 0 < x_1, x_2 < 1, (x_1 + x_2)/2 < c\}$,that is, if $c = 1/2$,

$$P[\ (X_1 + X_2) < 1\ ] = \int_0^1\int_0^{1-x_1} 4x_1x_2\ dx_2dx_1 = \int_0^1 2x_1(1 - x_1)^2\ dx_1 = \frac{1}{6}$$

EXAMPLE 3 Let X_1, X_2 be continuous random variables with ranges $\mathbb{X}_1 \equiv \mathbb{X}_2 \equiv [0,1]$, and joint pdf defined by

$$f_{X_1,X_2}(x_1,x_2) = 1 \qquad 0 \leq x_1, x_2 \leq 1$$

and zero otherwise. Let $Y = X_1 + X_2$. The has range $\mathbb{Y} \equiv [0,2]$,

$$F_Y(y) = P[Y \leq y] = P[(X_1 + X_2) \leq y]$$

Now, to calculate $P[(X_1 + X_2) \leq y]$, need to integrate f_{X_1,X_2} over the set

$$A_y = \{(x_1,x_2) : 0 < x_1, x_2 < 1, x_1 + x_2 \leq y\}$$

This region is a portion of the unit square (that is, $\mathbb{X}_1 \times \mathbb{X}_2$) ; the line $x_1 + x_2 = y$ is a line with negative slope that cuts the x_1 (horizontal) axis at $x_1 = y$, and the x_2 axis (vertical) at $x_2 = y$. Now for $0 \leq y \leq 1$, A_y is the dark shaded lower triangle in Figure 1(a); hence,for fixed y,

$$P[X_1 + X_2 < y] = \int_0^y \int_0^{y-x_2} 1 dx_1 dx_2 = \int_0^y (y - x_2) dx_2 = \frac{y^2}{2}.$$

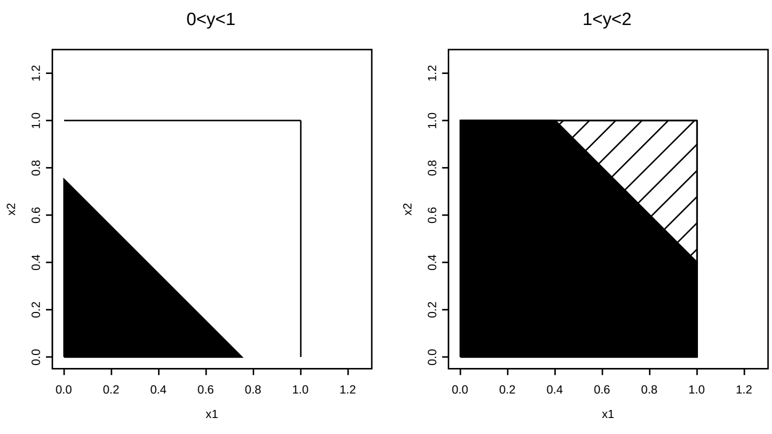
For $1 \leq y \leq 2$, A_y more complicated see Figure 1(b). It is easier mathematically to describe the complement of A_y within $\mathbb{X}_1 \times \mathbb{X}_2$ (striped in Figure 1(b)), so we instead compute the complement probability as follows:

$$P[X_1 + X_2 \leq y] = 1 - \int_{y-1}^1 \int_{y-x_2}^1 1 \, dx_1 dx_2 = 1 - \int_{y-1}^1 (1 - y + x_2) dx_2 = -\frac{y^2}{2} + 2y - 1$$

These two expressions give the cdf F_Y, and hence by differentiation we have

$$f_Y(y) = \begin{cases} y & 0 \leq y \leq 1 \\ \\ 2 - y & 1 \leq y \leq 2 \end{cases}$$

and zero otherwise.



EXAMPLE 4 Let X_1 and X_2 be continuous random variables with ranges $\mathbb{X}_1 = (0,1)$, $\mathbb{X}_2 = (0,2)$ and joint pdf defined by

$$f_{X_1,X_2}(x_1,x_2) = c\left(x_1^2 + \frac{x_1 x_2}{2}\right) \qquad 0 < x_1 < 1,\ 0 < x_2 < 2$$

and zero otherwise.

(i) To calculate c, we have

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)dx_1dx_2 &= \int_0^2 \left\{\int_0^1 c\left(x_1^2 + \frac{x_1x_2}{2}\right)dx_1\right\}dx_2 \\
&= \int_0^2 c\left[\frac{x_1^3}{3} + \frac{x_1^2x_2}{4}\right]_0^1 dx_2 \\
&= \int_0^2 c\left(\frac{1}{3} + \frac{x_2}{4}\right)dx_2 \\
&= c\left[\frac{x_2}{3} + \frac{x_2^2}{8}\right]_0^2 = c\frac{7}{6}
\end{aligned}$$

so $c = 6/7$. The marginal pdf of X_1 is given, for $0 < x_1 < 1$, by

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\ dx_2 = \int_0^2 \frac{6}{7}\left(x_1^2 + \frac{x_1x_2}{2}\right)\ dx_2 = \frac{6}{7}\left[x_1^2x_2 + \frac{x_1x_2^2}{4}\right]_0^2 = \frac{6x_1(2x_1+1)}{7}$$

and is zero otherwise.

(ii) To compute $P[\ X_1 > X_2\ ]$, let

$$A = \{\ (x_1,x_2)\ :\ 0 < x_1 < 1, 0 < x_2 < 2, x_2 < x_1\ \}$$

so that

$$\begin{aligned}
P[\ X_1 &> X_2\ ] = \int\int_A f_{X_1,X_2}(x_1,x_2)\ dx_2dx_1 \\
&= \int_0^1 \left\{\int_0^{x_1} \frac{6}{7}\left(x_1^2 + \frac{x_1x_2}{2}\right)\ dx_2\right\}dx_1 \\
&= \int_0^1 \left[x_1^2x_2 + \frac{x_1x_2^2}{4}\right]_0^{x_1}\ dx_1 \\
&= \int_0^1 \left(x_1^3 + \frac{x_1^3}{4}\right)\ dx_1 \\
&= \frac{6}{7}\left[\frac{5x_1^4}{16}\right]_0^1 \\
&= \frac{15}{56}
\end{aligned}$$

EXAMPLE 5 Let X_1, X_2 and X_3 be continuous random variables with joint ranges

$$\mathbb{X}^{(3)} = \{(x_1, x_2, x_3) : 0 < x_1 < x_2 < x_3 < 1\}$$

and joint pdf defined by

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = c \qquad 0 < x_1 < x_2 < x_3 < 1$$

and zero otherwise.

(i) To calculate c, integrate carefully over $\mathbb{X}^{(3)}$, that is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(x_1, x_2, x_3) \; dx_1 \; dx_2 \; dx_3 = 1$$

gives that

$$\int_0^1 \left\{ \int_0^{x_3} \left\{ \int_0^{x_2} c \; dx_1 \right\} \; dx_2 \right\} \; dx_3 = 1$$

Now

$$\int_0^1 \left\{ \int_0^{x_3} \left\{ \int_0^{x_2} c \; dx_1 \right\} \; dx_2 \right\} \; dx_3 = \int_0^1 \left\{ \int_0^{x_3} c x_2 \; dx_2 \right\} \; dx_3 = \int_0^1 \frac{c x_3^2}{2} \; dx_3 = \frac{c}{6}$$

and hence $c = 6$.

Also, for $0 < x_3 < 1$, f_{X_3} is given by

$$f_{X_3}(x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(x_1, x_2, x_3) \; dx_1 \; dx_2 = \int_0^{x_3} \left\{ \int_0^{x_2} 6 \; dx_1 \right\} \; dx_2 = \int_0^{x_3} 6 x_2 \; dx_2 = 3 x_3^2$$

and is zero otherwise. Similar calculations for X_1 and X_2 give

$$f_{X_1}(x_1) = 3(1 - x_1)^2 \qquad 0 < x_1 < 1$$

$$f_{X_2}(x_2) = 6 x_2 (1 - x_2) \qquad 0 < x_2 < 1$$

with both densities equal to zero outside of these ranges.

Furthermore, for the **joint marginal** of X_1 and X_2, we have

$$f_{X_1,X_2}(x_1, x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(x_1, x_2, x_3) \; dx_3 = \int_{x_2}^{1} 6 \; dx_3 = 6(1 - x_2) \qquad 0 < x_1 < x_2 < 1$$

and zero otherwise. Combining these results, we have, for example, for the conditional of X_1 given $X_2 = x_2$,

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_2}(x_2)} = \frac{1}{x_2} \qquad 0 < x_1 < x_2$$

and zero otherwise for **fixed** x_2. Now, we can calculate the expectation of X_1 either directly or using the *Law of Iterated Expectation*: we have

$$E_{f_{X_1}}[\; X_1 \;] = \int_{-\infty}^{\infty} x_1 f_{X_1}(x_1) \; dx_1 = \int_0^1 x_1 \; 3(1 - x_1)^2 \; dx_1 = \frac{1}{4}$$

or, alternatively,

$$E_{f_{X_1|X_2}}[\ X_1|X_2 = x_2\ ] = \int_{-\infty}^{\infty} x_1 f_{X_1|X_2}(x_1|x_2)\ dx_1 = \int_0^{x_2} x_1 \frac{1}{x_2}\ dx_1 = \frac{x_2}{2}$$

and hence by the law of iterated expectation

$$\begin{aligned} E_{f_{X_1}}[X_1] &= E_{f_{X_2}}\left[E_{f_{X_1|X_2}}[X_1|X_2 = x_2]\right] = \int_{-\infty}^{\infty} \left\{E_{f_{X_1|X_2}}[X_1|X_2 = x_2]\right\} f_{X_2}(x_2)dx_2 \\ &= \int_0^1 \frac{x_2}{2} 6x_2(1-x_2)dx_2 = \frac{1}{4} \end{aligned}$$

EXAMPLE 6 Let X_1, X_2 be continuous random variables with joint density f_{X_1,X_2} and let random variable Y be defined by $Y = g(X_1, X_2)$. To calculate the pdf of Y we could use the multivariate transformation theorem after defining another (dummy) variable Z as some function of X_1 and X_2, and consider the joint transformation $(X_1, X_2) \longrightarrow (Y, Z)$.

As a special case of the Theorem, consider defining $Z = X_1$. We have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,Z}(y,z)\ dz = \int_{-\infty}^{\infty} f_{Y|Z}(y|z) f_Z(z)\ dz = \int_{-\infty}^{\infty} f_{Y|X_1}(y|x_1) f_{X_1}(x_1)\ dx_1$$

as $f_{Y,Z}(y,z) = f_{Y|Z}(y|z)f_Z(z)$ by the chain rule for densities; $f_{Y|X_1}(y|x_1)$ is a univariate (conditional) pdf for Y given $X_1 = x_1$.

Now, **given** that $X_1 = x_1$, we have that $Y = g(x_1, X_2)$, that is, Y is a transformation of X_2 only. Hence the conditional pdf $f_{Y|X_1}(y|x_1)$ can be derived using single variable (rather than multivariate) transformation techniques. Specifically, if $Y = g(x_1, X_2)$ is a 1-1 transformation from X_2 to Y, then the inverse transformation $X_2 = g^{-1}(x_1, Y)$ is well defined, and by the transformation theorem

$$f_{Y|X_1}(y|x_1) \quad = \quad f_{X_2|X_1}(g^{-1}(x_1,y))\,|J(y;x_1)| \quad = \quad f_{X_2|X_1}(g^{-1}(x_1,y)|x_1) \left|\frac{\partial}{\partial t}\left\{g^{-1}(x_1,t)\right\}_{t=y}\right|$$

and hence

$$f_Y(y) = \int_{-\infty}^{\infty} \left\{ f_{X_2|X_1}(g^{-1}(x_1,y)|x_1) \left|\frac{\partial}{\partial t}\left\{g^{-1}(x_1,t)\right\}_{t=y}\right| \right\} f_{X_1}(x_1)dx_1$$

For example, if $Y = X_1X_2$, then $X_2 = Y/X_1$, and hence

$$\left|\frac{\partial}{\partial t}\left\{g^{-1}(x_1,t)\right\}_{t=y}\right| = \left|\frac{\partial}{\partial t}\left\{\frac{t}{x_1}\right\}_{t=y}\right| = |x_1|^{-1}$$

so

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_2|X_1}(y/x_1|x_1)\,|x_1|^{-1}\, f_{X_1}(x_1)dx_1.$$

The conditional density $f_{X_2|X_1}$ and/or the marginal density f_{X_1} may be zero on parts of the range of the integral. Alternatively, the **cdf** of Y is given by

$$F_Y(y) = P[\ Y \leq y\ ] = P[\ g(X_1,X_2) \leq y\ ] = \int\int_{A_y} f_{X_1,X_2}(x_1,x_2)\ dx_2 dx_1$$

where $A_y = \{\ (x_1, x_2)\ : g(x_1, x_2) \leq y\ \}$ so the cdf can be calculated by carefully identifying and intergrating over the set A_y.

EXAMPLE 7 Let X_1, X_2 be random variables with joint density f_{X_1,X_2} and let $g(X_1)$. Then

$$\begin{aligned}
E_{f_{X_1,X_2}}\left[g(X_1)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1) f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} g(x_1) f_{X_1|X_2}(x_1|x_2) f_{X_2}(x_2)dx_1\right\}dx_2 \\
&= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} g(x_1) f_{X_1|X_2}(x_1|x_2)dx_1\right\} f_{X_2}(x_2)dx_2 \\
&= E_{f_{X_2}}\left[E_{f_{X_1|X_2}}\left[g(X_1)|X_2 = x_2\right]\right] \\
&= E_{f_{X_1}}\left[g(X_1)\right]
\end{aligned}$$

by the law of iterated expectation.

EXAMPLE 8 Let X_1, X_2 be continuous random variables with joint pdf given by

$$f_{X_1,X_2}(x_1,x_2) = x_1 \exp\{-(x_1+x_2)\} \qquad x_1, x_2 > 0$$

and zero otherwise. Let $Y = X_1 + X_2$. Then by the Convolution Theorem,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, y-x_1)\ dx_1 = \int_0^y x_1 \exp\{-(x_1 + (y-x_1))\}\ dx_1 = \frac{y^2}{2}e^{-y} \qquad y > 0$$

and zero otherwise. Note that the integral range is 0 to y as the joint density f_{X_1,X_2} is only non-zero when both its arguments are positive, that is, when $x_1 > 0$ and $y - x_1 > 0$ for fixed y, or when $0 < x_1 < y$. It is straightforward to check that this density is a valid pdf.

EXAMPLE 9 Let X_1, X_2 be continuous random variables with joint pdf given by

$$f_{X_1,X_2}(x_1,x_2) = 2(x_1+x_2) \qquad 0 \le x_1 \le x_2 \le 1$$

and zero otherwise. Let $Y = X_1 + X_2$. Then by the Convolution Theorem,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, y-x_1)\ dx_1 = \begin{cases} \displaystyle\int_0^{y/2} 2y\ dx_1 & 0 \le y \le 1 \\[2ex] \displaystyle\int_{y-1}^{y/2} 2y\ dx_1 & 1 \le y \le 2 \end{cases}$$

and zero otherwise, as $f_{X_1,X_2}(x_1, y-x_1) = 2y$; this holds when both x_1 and $y-x_1$ lie in the interval $[0,1]$ with $x_1 \le y - x_1$ for fixed y, and zero otherwise. Clearly Y takes values on $\mathbb{Y} \equiv [0,2]$; for $0 \le y \le 1$, the constraints $0 \le x_1 \le y - x_1 \le 1$ imply that $0 \le 2x_1 \le y$, or $0 \le x_1 \le y/2$ (for fixed y); if $1 \le y \le 2$ the constraints imply $1 - y \le x_1 \le y/2$. Hence

$$f_Y(y) = \begin{cases} y^2 & 0 \le y \le 1 \\[2ex] y(2-y) & 1 \le y \le 2 \end{cases}$$

It is straightforward to check that this density is a valid pdf. The region of (X_1, Y) space on which the joint density $f_{X_1,X_2}(x_1, y-x_1)$ is **positive**; this region is the triangle with corners $(0,0), (1,2), (0,1)$.

EXAMPLE 10 Let X_1, X_2 be continuous random variables with joint pdf given by

$$f_{X_1,X_2}(x_1,x_2) = c \qquad 0 < x_1 < 1, x_1 < x_2 < x_1 + 1$$

and zero otherwise. To calculate c, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\ dx_2 dx_1 = \int_0^1 \int_{x_1}^{x_1+1} c\ dx_2 dx_1 = \int_0^1 c\,[x_2]_{x_1}^{x_1+1}\,dx_1 = \int_0^1 c\ dx_2 = c$$

so $c = 1$. The marginal pdf of X_1 is given by

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\ dx_2 = \int_{x_1}^{x_1+1} 1\ dx_2 = 1 \qquad 0 < x_1 < 1$$

and zero otherwise, and the marginal pdf for X_2 is given by

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\ dx_1 = \begin{cases} \displaystyle\int_0^{x_2} 1\ dx_1 & = x_2 & 0 < x_2 < 1 \\ \displaystyle\int_{x_2-1}^{1} 1\ dx_1 & = 2 - x_2 & 1 \le x_2 < 2 \end{cases}$$

and zero otherwise. Hence

$$\begin{aligned} E_{f_{X_1}}[\ X_1\ ] &= \int_{-\infty}^{\infty} x_1 f_{X_1}(x_1)\ dx_1 = \int_0^1 x_1\ dx_1 = \frac{1}{2} \\ Var_{f_{X_1}}[\ X_1\ ] &= \int_{-\infty}^{\infty} x_1^2 f_{X_1}(x_1)\ dx_1 - \left\{E_{f_{X_1}}[\ X_1\ ]\right\}^2 = \int_0^1 x_1^2\ dx_1 - \frac{1}{4} = \frac{1}{12} \end{aligned}$$

$$\begin{aligned} E_{f_{X_2}}[\ X_2\ ] &= \int_{-\infty}^{\infty} x_2 f_{X_2}(x_2)\ dx_2 = \int_0^1 x_2^2\ dx_2 + \int_1^2 x_2(2 - x_2)\ dx_2 \\ &= \frac{1}{3} - \left(1 - \frac{1}{3}\right) + \left(4 - \frac{8}{3}\right) = 1 \end{aligned}$$

$$\begin{aligned} Var_{f_{X_2}}[\ X_2\ ] &= \int_{-\infty}^{\infty} x_2^2 f_{X_2}(x_2)\ dx_2 - \left\{E_{f_{X_2}}[\ X_2\ ]\right\}^2 \\ &= \int_0^1 x_2^2 x_2\ dx_2 + \int_1^2 x_2^2(2 - x_2)\ dx_2 - 1 \\ &= \frac{1}{4} - \left(\frac{2}{3} - \frac{1}{4}\right) + \left(\frac{16}{3} - 4\right) - 1 = \frac{1}{6} \end{aligned}$$

The covariance and correlation of X_1 and X_2 are then given by

$$\begin{aligned}
Cov_{f_{X_1,X_2}}[\ X_1, X_2\ ] &= \left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1,x_2)\ dx_2\right\} dx_1 - E_{f_{X_1}}[\ X_1\ ]\ E_{f_{X_2}}[\ X_2\ ] \\
&= \int_0^1 \left\{\int_{x_1}^{x_1+1} x_1 x_2\ dx_2\right\} dx_1 - \frac{1}{2}.1 \\
&= \int_0^1 x_1 \left[\frac{x_2}{2}\right]_{x_1}^{x_1+1}\ dx_1\ -\ \frac{1}{2} \\
&= \int_0^1 \left(x_1^2 + \frac{x_1}{2}\right)\ dx_1\ -\ \frac{1}{2} \\
&= \left[\frac{x_1^3}{3} + \frac{x_1^2}{4}\right]_0^1 - \frac{1}{2} \\
&= \frac{7}{12} - \frac{1}{2} = \frac{1}{12}
\end{aligned}$$

and hence

$$Corr_{f_{X_1,X_2}}[\ X_1, X_2\ ] = \frac{Cov_{f_{X_1,X_2}}[\ X_1, X_2\ ]}{\sqrt{Var_{f_{X_1}}[\ X_1\ ]\ Var_{f_{X_2}}[\ X_2\ ]}} = \frac{1/12}{\sqrt{1/12}\sqrt{1/6}} = \frac{1}{\sqrt{2}}$$