WORKED EXAMPLES 1

TOTAL PROBABILITY AND BAYES THEOREM

Example 1 A biased coin (with probability of obtaining a Head equal to $p > 0$) is tossed repeatedly and independently until the first head is observed. Compute the probability that the first head appears at an **even** numbered toss.

SOLUTION: Define

- sample space Ω to be all possible infinite binary sequences of coin tosses
- event H_1 - head on **first** toss
- event E - first head on even numbered toss.

We want $P(E)$: using the Theorem of Total Probability, and the partition of Ω given by $\{H_1, H_1'\}$

$$P(E) = P(E|H_1)P(H_1) + P(E|H_1')P(H_1')$$

Now clearly, $P(E|H_1) = 0$ (given H_1, that a head appears on the first toss, E cannot occur) and also $P(E|H_1')$ can be seen to be given by

$$P(E|H_1') = P(E') = 1 - P(E)$$

(given that a head does **not** appear on the first toss, the required conditional probability is merely the probability that the sequence concludes after a further **odd** number of tosses, that is, the probability of E'). Hence $P(E)$ satisfies

$$P(E) = 0 \times p + (1 - P(E)) \times (1 - p) = (1 - p)(1 - P(E))$$

so that

$$P(E) = \frac{1-p}{2-p}.$$

Alternately, consider the partition of E into $E_1, E_2, ...$ where E_k is the event that the first head occurs on the $2k$th toss. Then $E = \bigcup_{k=1}^{\infty} E_k$, and

$$P(E) = P\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} P(E_k).$$

Now $P(E_k) = (1-p)^{2k-1}\, p$ (that is, $2k - 1$ tails, then a head), so

$$\begin{aligned}
P(E) &= \sum_{k=1}^{\infty} (1-p)^{2k-1}\, p \\
&= \frac{p}{1-p} \sum_{k=1}^{\infty} (1-p)^{2k} \\
&= \frac{p}{1-p} \frac{(1-p)^2}{1-(1-p)^2} \\
&= \frac{1-p}{2-p}
\end{aligned}$$

Example 2 Two players A and B are competing at a trivia quiz game involving a series of questions. On any individual question, the probabilities that A and B give the correct answer are α and β respectively, for all questions, with outcomes for different questions being independent. The game finishes when a player wins by answering a question correctly.

Compute the probability that A wins if

(a) A answers the first question (b) B answers the first question

SOLUTION: Define

- sample space Ω to be all possible infinite sequences of answers
- event A - A answers the first question
- event F - game ends after the first question
- event W - A wins.

We want

$$P(W|A) \qquad \text{and} \qquad P(W|A')$$

Using the Theorem of Total Probability, and the partition given by $\{F, F'\}$

$$P(W|A) = P(W|A \cap F)P(F|A) + P(W|A \cap F')P(F'|A).$$

Now, clearly

$$P(F|A) = P\,[\text{A answers first question correctly}] = \alpha \qquad\qquad P(F'|A) = 1 - \alpha$$

and $P(W|A \cap F) = 1$, but $P(W|A \cap F') = P(W|A')$, so that

$$P(W|A) = (1 \times \alpha) + \left(P(W|A') \times (1 - \alpha)\right). = \alpha + P(W|A')\,(1 - \alpha) \tag{1}$$

Similarly

$$P(W|A') = P(W|A' \cap F)P(F|A') + P(W|A' \cap F')P(F'|A').$$

We have

$$P(F|A') = P\,[\text{B answers first question correctly}] = \beta \qquad\qquad P(F'|A') = 1 - \beta$$

but $P(W|A' \cap F) = 0$. Finally $P(W|A' \cap F') = P(W|A)$, so that

$$P(W|A') = (0 \times \beta) + (P(W|A) \times (1 - \beta)) = P(W|A)\,(1 - \beta) \tag{2}$$

Solving (1) and (2) simultaneously gives, for (a) and (b)

$$P(W|A) = \frac{\alpha}{1 - (1 - \alpha)\,(1 - \beta)} \qquad\qquad P(W|A') = \frac{(1 - \beta)\,\alpha}{1 - (1 - \alpha)\,(1 - \beta)}$$

Note: recall, for any events E_1 and E_2 we have that

$$P\left(E_1'|E_2\right) = 1 - P\left(E_1|E_2\right)$$

but not necessarily that

$$P\left(E_1|E_2'\right) = 1 - P\left(E_1|E_2\right)$$

Example 3 Patients are recruited onto the two arms (0 - Control, 1-Treatment) of a clinical trial. The probability that an adverse outcome occurs on the control arm is p_0, and is p_1 for the treatment arm. Patients are allocated alternately onto the two arms in the sequence 010101..., and their outcomes are independent

What is the probability that the first adverse event occurs on the control arm. ?

SOLUTION: Define

- sample space Ω to be all possible infinite sequences of patients outcomes
- event E_1 - first patient (allocated onto the control arm) suffers an adverse outcome
- event E_2 - first patient (allocated onto the control arm) does not suffer an adverse outcome, but the second patient (allocated onto the treatment arm) does suffer an adverse outcome
- event E_0 - neither of the first two patients suffer adverse outcomes
- event F - first adverse event occurs on the control arm

We want $P(F)$. Now the events E_1, E_2 and E_0 partition Ω, so, by the Theorem,

$$P(F) = P(F|E_1)P(E_1) + P(F|E_2)P(E_2) + P(F|E_0)P(E_0).$$

Now

$$P(E_1) = p_0 \qquad P(E_2) = (1-p_0)\,p_1 \qquad P(E_0) = (1-p_0)\,(1-p_1)$$

and $P(F|E_1) = 1$, $P(F|E_2) = 0$. Finally, as after two non-adverse outcomes, the allocation process effectively re-starts, so $P(F|E_0) = P(F)$. Hence

$$P(F) = (1 \times p_0) + (0 \times (1-p_0)\,p_1) + (P(F) \times (1-p_0)\,(1-p_1)) = p_0 + (1-p_0)\,(1-p_1)\,P(F)$$

which can be re-arranged to give

$$P(F) = \frac{p_0}{p_0 + p_1 - p_0 p_1}$$

Example 4 In a tennis match, with the score at deuce, the game is one by the first player who gets a clear lead of two points.

If the probability that given player wins a particular point is θ, and all points are played independently, what is the probability that player eventually wins the game

SOLUTION: Define

- sample space Ω to be all possible infinite sequences of points
- event W_i - nominated player wins the ith point
- event V_i - nominated player wins the game on the ith point
- event V - nominated player wins the game.

We want $P(V)$. The events $\{W_1, W_1'\}$ partition Ω, and thus, by the Theorem

$$P(V) = P(V|W_1)P(W_1) + P(V|W_1')P(W_1'). \tag{3}$$

Now $P(W_1) = \theta$ and $P(W_1') = 1-\theta$. To get $P(V|W_1)$ and $P(V|W_1')$, we need to further condition on the result of the second point, and again use the Theorem: for example

$$\begin{aligned} P(V|W_1) &= P(V|W_1 \cap W_2)P(W_2|W_1) + P(V|W_1 \cap W_2')P(W_2'|W_1) \\ P(V|W_1') &= P(V|W_1' \cap W_2)P(W_2|W_1') + P(V|W_1' \cap W_2')P(W_2'|W_1') \end{aligned} \tag{4}$$

where

$$\begin{array}{ll} P(V|W_1 \cap W_2) = 1 & P(W_2|W_1) = P(W_2) = \theta \\ P(V|W_1 \cap W_2') = P(V) & P(W_2'|W_1) = P(W_2') = 1-\theta \\ P(V|W_1' \cap W_2) = P(V) & P(W_2|W_1') = P(W_2) = \theta \\ P(V|W_1' \cap W_2') = 0 & P(W_2'|W_1') = P(W_2') = 1-\theta \end{array}$$

as,

given $W_1 \cap W_2$	:	the game is over, and the player has won
given $W_1 \cap W_2'$	:	the game is back at deuce
given $W_1' \cap W_2$	:	the game is back at deuce
given $W_1' \cap W_2'$	:	the game is over, and the player has lost

and the results of successive points are independent. Thus

$$\begin{aligned} P(V|W_1) &= (1 \times \theta) + (P(V) \times (1-\theta)) = \theta + (1-\theta)\,P(V) \\ P(V|W_1') &= (P(V) \times \theta) + 0 \times (1-\theta) = \theta P(V) \end{aligned}$$

Hence, combining (3) and (4) we have

$$P(V) = (\theta + (1-\theta)\,P(V))\,\theta + \theta P(V)\,(1-\theta) = \theta^2 + 2\theta(1-\theta)P(V) \Longrightarrow P(V) = \frac{\theta^2}{1-2\theta(1-\theta)}$$

Alternately, $\{V_i, i = 1, 2, ...\}$ partition V. Hence

$$P(V) = \sum_{i=1}^{\infty} P(V_i)$$

Now, $P(V_i) = 0$ if i is odd, as the game can never be completed after an odd number of points. For $i = 2$, $P(V_2) = \theta^2$, and for $i = 2k+2$ $(k = 1, 2, 3, ...)$ we have

$$P(V_i) = P(V_{2k+2}) = 2^k \theta^k (1-\theta)^k \times \theta^2$$

- the score must stand at deuce after $2k$ points and the game must not have been completed prior to this, indicating that there must have been k successive drawn pairs of points, each of which could be arranged win/lose or lose/win for the nominated player. Then that player must win the final two points. Hence

$$P(V) = \sum_{k=0}^{\infty} P(V_{2k+2}) = \theta^2 \sum_{k=0}^{\infty} \{2\theta(1-\theta)\}^k = \frac{\theta^2}{1-2\theta(1-\theta)}$$

as the term in the geometric series satisfies $|2\theta(1-\theta)| < 1$.

Example 5 A coin for which $P(\text{Heads}) = p$ is tossed until two successive Tails are obtained.

Find the probability that the experiment is completed on the nth toss.

SOLUTION: Define

- sample space Ω to be all possible infinite sequences of tosses
- event E_1 : first toss is H
- event E_2 : first two tosses are TH
- event E_3 : first two tosses are TT
- event F_n : experiment completed on the nth toss

We want $P(F_n)$ for $n = 2, 3, ...$ .The events $\{E_1, E_2, E_3\}$ partition Ω, and thus, by the Theorem

$$P(F_n) = P(F_n|E_1)P(E_1) + P(F_n|E_2)P(E_2) + P(F_n|E_3)P(E_3). \tag{5}$$

Now for $n = 2$

$$P(F_2) = P(E_3) = (1-p)^2$$

and for $n > 2$,

$$P(F_n|E_1) = P(F_{n-1}) \qquad P(F_n|E_2) = P(F_{n-2}) \qquad P(F_n|E_3) = 0$$

as given E_1 we need $n-1$ further tosses that finish TT for F_n to occur, and given E_2, we need $n-2$ further tosses that finish TT for F_n to occur, with all tosses independent. Hence, if $p_n = P(F_n)$ then $p_2 = (1-p)^2$, otherwise, from (5), p_n satisfies

$$p_n = (p_{n-1} \times p) + (p_{n-2} \times (1-p)\, p) = pp_{n-1} + p(1-p)p_{n-2}$$

To find p_n explicitly, try a solution of the form $p_n = A\lambda_1^n + B\lambda_2^n$ which gives

$$A\lambda_1^n + B\lambda_2^n = p\left(A\lambda_1^{n-1} + B\lambda_2^{n-1}\right) + p(1-p)\left(A\lambda_1^{n-2} + B\lambda_2^{n-2}\right).$$

First, collecting terms in λ_1 gives

$$\lambda_1^n = p\lambda_1^{n-1} + p(1-p)\lambda_1^{n-2} \Longrightarrow \lambda_1^2 - p\lambda_1 - p(1-p) = 0$$

indicating that λ_1 and λ_2 are given as the roots of this quadratic, that is

$$\lambda_1 = \frac{p - \sqrt{p^2 + 4p(1-p)}}{2} \qquad \lambda_2 = \frac{p + \sqrt{p^2 + 4p(1-p)}}{2}$$

Furthermore,

$$\begin{aligned} n = 1 : p_1 = 0 \quad & \Longrightarrow A\lambda_1 + B\lambda_2 = 0 \\ n = 2 : p_2 = (1-p)^2 \quad & \Longrightarrow A\lambda_1^2 + B\lambda_2^2 = (1-p)^2 \end{aligned}$$

$$\Longrightarrow \quad A = \frac{(1-p)^2}{\lambda_1(\lambda_1 - \lambda_2)} \qquad B = -\frac{\lambda_1}{\lambda_2}A = -\frac{(1-p)^2}{\lambda_2(\lambda_1 - \lambda_2)}$$

$$\Longrightarrow \quad p_n = -\frac{(1-p)^2\lambda_1^n}{\lambda_1(\lambda_2 - \lambda_1)} + \frac{(1-p)^2\lambda_2^n}{\lambda_2(\lambda_2 - \lambda_1)} = \frac{(1-p)^2}{\sqrt{p(4-3p)}}\left(\lambda_2^{n-1} - \lambda_1^{n-1}\right) \qquad n \geq 2$$