GENERATING FUNCTIONS FACTSHEET

For random variable X with mass/density function f_X, the **moment generating function**, or mgf, of X, M_X, is defined by

$$M_X(t) = E_{f_X}[e^{tX}]$$

if this expectation exists for all values of $t \in (-h, h)$ for some $h > 0$, that is,

$$M_X(t) = \begin{cases} \sum e^{tx} f_X(x) & X \text{ Discrete} \\ \\ \int e^{tx} f_X(x)dx & X \text{ Continuous} \end{cases}$$

where the sum/integral is over $\mathbb{X}$.

KEY PROPERTIES OF MGFS

1. There is a **1-1 correspondence between generating functions and distributions**: if X_1 and X_2 are random variables taking values on X with mass/density functions f_{X_1} and f_{X_2}, and mgfs M_{X_1} and M_{X_2} respectively, then

$$f_{X_1}(x) \equiv f_{X_2}(x), x \in X \iff M_{X_1}(t) \equiv M_{X_2}(t), t \in (-h, h)$$

2. If X is a discrete random variable, the rth derivative of M_X evaluated at t, $M_X^{(r)}(t)$, is given by

$$M_X^{(r)}(t) = \frac{d^r}{ds^r}\left\{M_X(s)\right\}_{s=t} = \frac{d^r}{ds^r}\left\{\sum e^{sx} f_X(x)\right\}_{s=t} = \sum x^r e^{tx} f_X(x)$$

and hence

$$M_X^{(r)}(0) = \sum x^r f_X(x) = E_{f_X}[X^r]$$

Similarly, if X is a continuous random variable, the rth derivative of M_X is given by

$$M_X^{(r)}(t) = \frac{d^r}{ds^r}\left\{\int e^{sx} f_X(x) d_x\right\}_{s=t} = \int x^r e^{tx} f_X(x)dx$$

and hence

$$M_X^{(r)}(0) = \int x^r f_X(x)dx = E_{f_X}[X^r]$$

3. If X is a discrete random variable, then

$$M_X(t) = \sum e^{tx} f_X(x) = \sum\left\{\sum_{r=0}^{\infty}\frac{(tx)^r}{r!}\right\} f_X(x) = 1 + \sum_{r=1}^{\infty}\frac{t^r}{r!}\left\{\sum x^r f_X(x)\right\} = 1 + \sum_{r=1}^{\infty}\frac{t^r}{r!}E_{f_X}[X^r]$$

The identical result holds in the continuous case.

4. From the general result for expectations of functions of random variables

$$M_Y(t) = E_{f_Y}[e^{tY}] = E_{f_X}[e^{t(aX+b)}] = E_{f_X}[e^{t(aX+b)}] = e^{bt}E_{f_X}[e^{atX}] = e^{bt}M_X(at).$$

Therefore, if $Y = aX + b$,

$$M_Y(t) = e^{bt}M_X(at)$$

5. Let $X_1, ..., X_k$ be independent random variables with mgfs $M_{X_1}, ..., M_{X_k}$ respectively. Then if random variable Y is defined by $Y = X_1 + ... + X_k$,

$$M_Y(t) = \prod_{i=1}^{k} M_{X_i}(t)$$

To see this for $k = 2$, consider X_1 and X_2 independent, integer-valued, discrete r.v.s, then if $Y = X_1 + X_2$, by the **Theorem of Total Probability**, partitioning the event $Y = y$ as follows

$$Y = y \equiv \bigcup_{x_1}(Y = y \cap X_1 = x_1) \equiv \bigcup_{x_1}(X_1 = x_1 \cap X_2 = y - x_1),$$

so by Axiom 3 and the independence assumption

$$\begin{aligned} f_Y(y) = P[Y = y] = \sum_{x_1} P[X_1 = x_1 \cap X_2 = y - x_1] &= \sum_{x_1} P[X_1 = x_1]P[X_2 = y - x_1] \\ &= \sum_{x_1} f_{X_1}(x_1) f_{X_2}(y - x_1). \end{aligned}$$

Hence

$$\begin{aligned} M_Y(t) &= E_{f_Y}[e^{tY}] = \sum_y e^{ty} f_Y(y) = \sum_y e^{ty} \left\{ \sum_{x_1} f_{X_2}(y - x_1) f_{X_2}(x_1) \right\} \\ &= \sum_{x_2} e^{t(x_1+x_2)} \left\{ \sum_{x_1} f_{X_2}(x_2) f_{X_2}(x_1) \right\} \qquad \text{(changing variables to } x_2 = y - x_1) \\ &= \left\{ \sum_{x_1} e^{tx_1} f_{X_1}(x_1) \right\} \left\{ \sum_{x_2} e^{tx_2} f_{X_2}(x_2) \right\} = M_{X_1}(t) M_{X_2}(t) \qquad . \end{aligned}$$

The result follows for general k by recursion. The result for continuous random variables follows in the obvious way. If $X_1, ..., X_k$ are **identically distributed**, then $M_{X_i}(t) \equiv M_X(t)$, say, for all i, so

$$M_Y(t) = \prod_{i=1}^{k} M_X(t) = \{M_X(t)\}^k$$

6. For random variable X, with mass/density function f_X, the **factorial moment** (fmgf) or **probability generating function** (pgf), of X, denoted G_X, is defined by

$$G_X(t) = E_{f_X}[t^X] = E_{f_X}[e^{X \log t}] = M_X(\log t)$$

if this expectation exists for all values of $t \in (1 - h, 1 + h)$ for some $h > 0$. Note

$$G_X^{(r)}(t) = \frac{d^r}{ds^r} \{G_X(s)\}_{s=t} = E_{f_X}\left[X(X-1)...(X-r+1)t^{X-r}\right]$$

$$\therefore G_X^{(r)}(1) = E_{f_X}[X(X-1)...(X-r+1)]$$

where $\mathrm{E}_{f_X}[X(X-1)...(X-r+1)]$ is the r**th factorial moment**. For discrete random variables, it can be shown by using a Taylor series expansion of G_X that, for $r = 1, 2, ...,$

$$\frac{G_X^{(r)}(0)}{r!} = P[X = r]$$