M2S1 - EXERCISES 7

The Central Limit Theorem

1. Using the Central Limit Theorem, construct Normal approximations to probability distribution of a random variable X having

(i) a Binomial distribution, $X \sim Binomial(n, \theta)$

(ii) a Poisson distribution, $X \sim Poisson(\lambda)$

(iii) a Negative Binomial distribution, $X \sim NegBinomial(n, \theta)$

(iv) a Gamma distribution, $X \sim Gamma(\alpha, \beta)$ *(Recall the additivity property for independent Gamma random variables, and note that* $\alpha = \sum_{i=1}^{n} (\alpha/n)$*).*

Extreme Order Statistics And Limiting Distributions

The following questions concern extreme **order statistics**; if $X_1, ...X_n$ are a collection of independent and identically distributed random variables taking values on $\mathbb{X}$ with mass function/pdf f_X and cdf F_X, let Y_n and Z_n correspond to the *maximum* and *minimum* order statistics derived from $X_1, ...X_n$, that is

$$Y_n = \max\{X_1, ..., X_n\} \qquad Z_n = \min\{X_1, ..., X_n\}.$$

The cdfs of Y_n and Z_n are given (please verify this yourself by considering $P[Y_n \leq y]$ and $P[Z_n > z]$) by

$$F_{Y_n}(y) = \{F_X(y)\}^n \qquad F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n.$$

2. Suppose $X_1, ..., X_n \sim Uniform(0, 1)$, that is

$$F_X(x) = x \qquad 0 \leq x \leq 1$$

Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$.

3. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = 1 - x^{-1} \qquad x \geq 1$$

Find the cdfs of Z_n and $U_n = Z_n^n$, and the limiting distributions of Z_n and U_n as $n \longrightarrow \infty$.

4. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = \frac{1}{1 + e^{-x}} \qquad x \in \mathbb{R}$$

Find the cdfs of Y_n and $U_n = Y_n - \log n$ and the limiting distributions of Y_n and U_n as $n \longrightarrow \infty$.

5. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = 1 - \frac{1}{1 + \lambda x} \qquad x > 0$$

Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$. Find also the cdfs of $U_n = Y_n/n$ and $V_n = nZ_n$, and the limiting distributions of U_n and V_n as $n \longrightarrow \infty$.

6. *Convergence in Probability*: Suppose $X_1, ..., X_n \sim Poisson(\lambda)$. Let

$$M_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Show that $M_n \xrightarrow{p} \lambda$ as $n \longrightarrow \infty$. If random variable T_n is defined by $T_n = e^{-M_n}$, show that $T_n \xrightarrow{p} e^{-\lambda}$, and using the Central Limit Theorem find the approximate probability distribution of T_n as $n \longrightarrow \infty$.