M2S1 - ASSESSED COURSEWORK 1

To be handed in no later than Wednesday, 2nd November, 12.00pm.
Please hand in to the Mathematics General Office

(a) The number of perforations successfully stamped into a 1 cm square of material used in a filter is a discrete random variable, N, with probability mass function (pmf) f_N defined by

$$f_N(n) = k_1 \frac{(-\log(1-\phi))^n}{n!} \qquad n = 0, 1, 2, \ldots$$

and zero otherwise, for some parameter ϕ, and constant k_1. (Note: as always, log means $\log_e$ or ln)

(i) State the range of values that ϕ can take in order for this to be a valid pmf.

[1 MARK]

(ii) Find an expression for k_1 as a function of ϕ.

[2 MARKS]

(iii) Find an expression for $P[N > 0]$.

[2 MARKS]

(b) The amount of liquid that flows through a single perforation in unit time is also a random variable, X, with the probability density function (pdf) f_X given by

$$f_X(x) = k_2 x \exp\{-\beta x\} \qquad x > 0$$

and zero otherwise, for parameter $\beta > 0$, and constant k_2.

(i) Find an expression for k_2 as a function of β.

[1 MARK]

(ii) Find the cumulative distribution function of X, F_X.

[2 MARKS]

(iii) Find an expression for $P[X > x]$.

[2 MARKS]

(c) Suppose that the flows through different perforations are independent stochastic phenomena, so that the random variables $X_1, X_2, \ldots$ corresponding to the flows through perforations labelled $1, 2, \ldots$ are independent, and have the same probability distribution as X from (b).

The total flow through the filter is the sum of the flows through the individual perforations. Let T denote the total flow through a 1cm square of filter material in unit time. By using the partition of the event $(T \leq t)$ for $t > 0$ given by

$$(T \leq t) \equiv \bigcup_{n=0}^{\infty} (N = n \cap T \leq t).$$

and elementary properties of moment generating functions, find expressions for the expectation and variance of T.

Hint: if we condition on $N = n$, and $n > 0$, then T can be represented as the sum of n independent and identically distributed random variables $X_1, X_2, \ldots, X_n$.

[10 MARKS]