EXPECTATION AND GENERATING FUNCTION CALCULATIONS FOR STANDARD DISTRIBUTIONS

DISCRETE PROBABILITY DISTRIBUTIONS

BERNOULLI DISTRIBUTION

MASS FUNCTION

$$f_X(x) = \theta^x (1-\theta)^{1-x} \qquad \text{for } x \in \{0,1\} \qquad \text{where } 0 \le \theta \le 1.$$

MGF

$$M_X(t) \;=\; \sum_{x=0}^{1} \mathrm{e}^{tx} \theta^x (1-\theta)^{1-x} \;= 1 - \theta + \theta \mathrm{e}^t$$

rth MOMENT

$$M_X^{(r)}(t) = \theta \mathrm{e}^t \Longrightarrow M_X^{(r)}(0) = \theta \qquad r = 1, 2, \ldots$$

$$\Longrightarrow \mathrm{E}_{f_X}[X] = \theta \qquad \mathrm{E}_{f_X}[X^2] = \theta \Longrightarrow \mathrm{Var}_{f_X}[X] = \;\theta - \theta^2 \;=\; \theta(1-\theta)$$

BINOMIAL DISTRIBUTION

MASS FUNCTION

$$f_X(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \qquad \text{for } x \in \{0,1,2,\ldots,n\} \qquad \text{where } n \ge 0,\; 0 \le \theta \le 1.$$

MGF

$$M_X(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} \theta^x (1-\theta)^{n-x} \;=\; \sum_{x=0}^{n} \binom{n}{x} \left(\theta e^t\right)^x (1-\theta)^{n-x} = \left(1 - \theta + \theta e^t\right)^n$$

rth MOMENT

No simple general expression for $M_X^{(r)}(t)$, but

$$M_X^{(1)}(t) = n\theta \mathrm{e}^t \left(1 - \theta + \theta \mathrm{e}^t\right)^{n-1} \qquad M_X^{(2)}(t) = n(n-1)\left\{\theta \mathrm{e}^t\right\}^2 \left(1 - \theta + \theta \mathrm{e}^t\right)^{n-2} + n\theta \mathrm{e}^t \left(1 - \theta + \theta \mathrm{e}^t\right)^{n-1}$$

so that $M_X^{(1)}(0) = n\theta$ and $M_X^{(2)}(0) = n(n-1)\theta^2 + n\theta$, and thus

$$\mathrm{E}_{f_X}[X] = \; n\theta \qquad \mathrm{Var}_{f_X}[X] = \; n(n-1)\theta^2 + n\theta - n^2\theta^2 = n\theta(1-\theta)$$

POISSON DISTRIBUTION

MASS FUNCTION

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad \text{for } x \in \{0, 1, 2, ...\} \qquad \text{where } \lambda > 0.$$

MGF

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = \exp\left\{\lambda\left(e^t - 1\right)\right\}$$

rth MOMENT

No simple general expression for $M_X^{(r)}(t)$, but

$$M_X^{(1)}(t) = \lambda e^t \exp\left\{\lambda\left(e^t - 1\right)\right\} \qquad M_X^{(2)}(t) = \left(\lambda e^t\right)^2 \exp\left\{\lambda\left(e^t - 1\right)\right\} + \lambda e^t \exp\left\{\lambda\left(e^t - 1\right)\right\}$$

so that $M_X^{(1)}(0) = \lambda$ and $M_X^{(2)}(0) = \lambda^2 + \lambda$, and thus

$$E_{f_X}[X] = \lambda \qquad \text{Var}_{f_X}[X] = \lambda^2 + \lambda - \lambda^2 = \lambda$$

GEOMETRIC DISTRIBUTION

MASS FUNCTION

$$f_X(x) = (1-\theta)^{x-1}\theta \qquad \text{for } x \in \{1, 2, ...\} \qquad \text{where } 0 \le \theta \le 1.$$

MGF

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx}(1-\theta)^{x-1}\theta = \theta e^t \sum_{x=1}^{\infty} e^{t(x-1)}(1-\theta)^{x-1} = \theta e^t \sum_{x=0}^{\infty} \left(e^t(1-\theta)\right)^x = \frac{\theta e^t}{1 - e^t(1-\theta)}$$

rth MOMENT

No simple general expression for $M_X^{(r)}(t)$, but

$$M_X^{(1)}(t) = \frac{\theta e^t}{\left[1 - e^t(1-\theta)\right]^2} \qquad M_X^{(2)}(t) = \frac{\theta e^t \left[1 - e^t(1-\theta)\right]\left[1 + e^t(1-\theta)\right]}{\left[1 - e^t(1-\theta)\right]^4}$$

so that $M_X^{(1)}(0) = \dfrac{1}{\theta}$ and $M_X^{(2)}(0) = \dfrac{2-\theta}{\theta^2}$, and thus

$$E_{f_X}[X] = \frac{1}{\theta} \qquad \text{Var}_{f_X}[X] = \frac{2-\theta}{\theta^2} - \frac{1}{\theta^2} = \frac{1-\theta}{\theta^2}$$

NEGATIVE BINOMIAL DISTRIBUTION

MASS FUNCTION

$$f_X(x) = \binom{x-1}{n-1} \theta^n (1-\theta)^{x-n} \qquad \text{where } n \in \{1,2,3,\ldots\}\,,\ 0 \le \theta \le 1.$$

MGF

$$M_X(t) = \sum_{x=n}^{\infty} \mathrm{e}^{tx} \binom{x-1}{n-1} \theta^n (1-\theta)^{x-n} = \left(\theta \mathrm{e}^t\right)^n \sum_{x=n}^{\infty} \binom{x-1}{n-1} \left(\mathrm{e}^t(1-\theta)\right)^{x-n} = \left\{\frac{\theta \mathrm{e}^t}{1-\mathrm{e}^t(1-\theta)}\right\}^n$$

rth MOMENT

No simple general expression for $M_X^{(r)}(t)$, but

$$M_X^{(1)}(t) = \frac{n(\theta \mathrm{e}^t)^n}{[1-\mathrm{e}^t(1-\theta)]^{n+1}} \qquad M_X^{(2)}(t) = \frac{n(\theta \mathrm{e}^t)^n\left[n + \mathrm{e}^t(1-\theta)\right]}{[1-\mathrm{e}^t(1-\theta)]^{n+2}}$$

so that $M_X^{(1)}(0) = \dfrac{n}{\theta}$ and $M_X^{(2)}(0) = \dfrac{n(n+(1-\theta))}{\theta^2}$, and thus

$$\mathrm{E}_{f_X}[X] = \frac{n}{\theta} \qquad \mathrm{Var}_{f_X}[X] = \frac{n(n+(1-\theta))}{\theta^2} - \frac{n^2}{\theta^2} = \frac{n(1-\theta)}{\theta^2}$$

CONTINUOUS PROBABILITY DISTRIBUTIONS

CONTINUOUS UNIFORM DISTRIBUTION

PDF

$$f_X(x) = \frac{1}{b-a} \qquad a \leq x \leq b$$

rth MOMENT

$$\mathrm{E}_{f_X}[X^r] \;=\; \int_a^b x^r \frac{1}{b-a}\,dx \;=\; \frac{1}{b-a}\left[\frac{b^{r+1}}{r+1} - \frac{a^{r+1}}{r+1}\right]$$

so therefore

$$\mathrm{E}_{f_X}[X] \;=\; \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right] \;=\; \frac{(a+b)}{2}$$

$$\mathrm{E}_{f_X}[X^2] \;=\; \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right] \;=\; \frac{(a^2+ab+b^2)}{3}$$

$$\Longrightarrow \mathrm{Var}_{f_X}[X] \;=\; \frac{(a^2+ab+b^2)}{3} - \left\{\frac{(a+b)}{2}\right\}^2$$

$$= \frac{(a-b)^2}{12}$$

EXPONENTIAL DISTRIBUTION

PDF

$$f_X(x) = \lambda \mathrm{e}^{-\lambda x} \qquad x > 0 \qquad \text{where } \lambda > 0.$$

MGF

$$M_X(t) \;=\; \int_0^\infty \mathrm{e}^{tx}\lambda\mathrm{e}^{-\lambda x}\,dx = \lambda\int_0^\infty \mathrm{e}^{-(\lambda-t)x}\,dx = \frac{\lambda}{\lambda-t} \qquad \text{for } t < \lambda$$

rth MOMENT

$$M_X^{(r)}(t) = \frac{r!\lambda}{(\lambda-t)^{r+1}} \Longrightarrow M_X^{(r)}(0) = \frac{r!}{\lambda^r}$$

and therefore, evaluating for $r = 1$ and 2, we have

$$\mathrm{E}_{f_X}[X] = \frac{1}{\lambda},\ \mathrm{E}_{f_X}[X^2] = \frac{2}{\lambda^2} \Longrightarrow \mathrm{Var}_{f_X}[X] = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \;=\; \frac{1}{\lambda^2}$$

GAMMA DISTRIBUTION

PDF

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\beta x} \qquad x > 0 \qquad \text{where } \alpha, \beta > 0.$$

MGF

$$M_X(t) = \int_0^\infty \mathrm{e}^{tx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\beta x}\, dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1}\, \mathrm{e}^{-(\beta-t)x}\, dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta-t)^\alpha} = \left(\frac{\beta}{\beta-t}\right)^\alpha$$

as the integrand is a pdf (of a $Gamma(\alpha, \beta - t)$ random variable).

rth MOMENT

$$M_X^{(1)}(t) = \frac{\alpha\beta^\alpha}{(\beta-t)^{\alpha+1}} \quad M_X^{(2)}(t) = \frac{\alpha(\alpha+1)\beta^\alpha}{(\beta-t)^{\alpha+2}} \Longrightarrow M_X^{(1)}(0) = \frac{\alpha}{\beta} \qquad M_X^{(2)}(0) = \frac{\alpha(\alpha+1)}{\beta^2}$$

and thus

$$\mathrm{E}_{f_X}[X] = \frac{\alpha}{\beta} \quad \mathrm{Var}_{f_X}[X] = \frac{\alpha(\alpha+1)}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$$

STANDARD NORMAL DISTRIBUTION

PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\} \qquad x \in \mathbb{R}$$

MGF

$$\begin{aligned} M_X(t) &= \int_{-\infty}^\infty \mathrm{e}^{tx} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left\{-\frac{1}{2}x^2 + tx\right\} dx \\ &= \exp\left\{\frac{t^2}{2}\right\} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-t)^2\right\} dx = \exp\left\{\frac{t^2}{2}\right\} \end{aligned}$$

by first completing the square in x, and then by noting that the integrand is a pdf (of an $N(t, 1)$ variable), and thus the integral is equal to one.

Hence we can write down the mgf of a non-standard normal random variable. If $X \sim N(0, 1)$, then $Y = \sigma X + \mu \sim N(\mu, \sigma^2)$ from previous transformation results. Hence by the standard mgf result, we have

$$M_Y(t) = \mathrm{E}_{f_Y}[e^{tY}] = \mathrm{E}_{f_X}\left[e^{(\sigma X+\mu)t}\right] = e^{\mu t}\mathrm{E}_{f_X}\left[e^{\sigma t X}\right] = e^{\mu t} M_X(\sigma t) = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$$

rth MOMENT

No simple general expression for $M_X^{(r)}(t)$, but

$$M_X^{(1)}(t) = (\mu + t\sigma^2)\exp\left\{\mu t + \frac{t^2\sigma^2}{2}\right\} \qquad M_X^{(2)}(t) = (\mu^2 + 2t\sigma^2\mu + t^2\sigma^4 + \sigma^2)\exp\left\{\mu t + \frac{t^2\sigma^2}{2}\right\}$$

so that $M_X^{(1)}(0) = \mu$ and $M_X^{(2)}(0) = \mu^2 + \sigma^2$, and thus

$$\mathrm{E}_{f_X}[X] = \mu \qquad \mathrm{Var}_{f_X}[X] = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

EXPECTATIONS AND VARIANCES OF STANDARD DISTRIBUTIONS

	Parameters	EXPECTATION	VARIANCE
Discrete Distributions			
$Bernoulli(\theta)$	θ	θ	$\theta(1-\theta)$
$Binomial(n,\theta)$	n,θ	$n\theta$	$n\theta(1-\theta)$
$Poisson(\lambda)$	λ	λ	λ
$Geometric(\theta)$	θ	$\frac{1}{\theta}$	$\frac{(1-\theta)}{\theta^2}$
$NegBinomial(n,\theta)$	n,θ	$\frac{n}{\theta}$	$\frac{n(1-\theta)}{\theta^2}$

Continuous Distributions

$Uniform(a,b)$	a,b	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$Exponential(\lambda)$	λ	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$Gamma(\alpha,\beta)$	α,β	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
$Normal(\mu,\sigma^2)$	μ,σ^2	μ	σ^2

MOMENT GENERATING FUNCTIONS OF STANDARD DISTRIBUTIONS

	Parameters	MGF
Discrete Distributions		
$Bernoulli(\theta)$	θ	$1-\theta+\theta\mathrm{e}^t$
$Binomial(n,\theta)$	n,θ	$(1-\theta+\theta\mathrm{e}^t)^n$
$Poisson(\lambda)$	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	θ	$\dfrac{\theta\mathrm{e}^t}{1-\mathrm{e}^t(1-\theta)}$
$NegBinomial(n,\theta)$	n,θ	$\left\{\dfrac{\theta\mathrm{e}^t}{1-\mathrm{e}^t(1-\theta)}\right\}^n$
Continuous Distributions		
$Exponential(\lambda)$	λ	$\dfrac{\lambda}{\lambda-t}$
$Gamma(\alpha,\beta)$	α,β	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Normal(\mu,\sigma^2)$	μ,σ^2	$\exp\left\{\mu t+\dfrac{\sigma^2t^2}{2}\right\}$