Statistical Inference and Methods

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Part III

Session 4: Multiple Time Series

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- Long-memory
- Cointegration
- Vector Autoregression

Long Memory

Persistence Process $\{X_t\}$ with acvs $\{\gamma_k\}$

- exhibits **long-memory** if the acvs is absolutely divergent

$$\sum_k |\gamma_k| = \infty$$

- exhibits **long-range dependence** if, $\forall a > 0$

$$\lim_{k \to \infty} \frac{a^{-k}}{\gamma_k} = 0$$

  that is, the acf is slowly decaying.

- in practice, diagnosed by observing large autocorrelation at high lags, spectral power near frequency zero.

Constructing Persistent Processes

Let $\{W_t\}$ be an i.i.d. Gaussian sequence with variance 1. Let $\delta \in (-1/2, 1/2)$.

- write

$$(1-B)^{\delta} = \sum_{k=0}^{\infty} c_k(-\delta)(-B)^k \qquad c_k(d) = \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)}$$

- Set $X_t = (1-B)^{-\delta} W_t$
- $\delta = 0$ gives i.i.d. sequence; $\delta = 1$ gives random walk.
- $-1/2 < \delta < 1/2$ gives fractional white noise

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Stationarity This *fractional differencing* yields a process that is

- **stationary** if $\delta < 1/2$
- **long-memory** if $0 < \delta < 1/2$.
- **long-range dependent** if $-1/2 < \delta < 1/2$.

For k large,

$$\gamma_k \sim \frac{1}{k^{1-2\delta}}$$

Seasonal Persistence Similar construction: replace $\{c_k\}$ sequence by $\{g_k\}$ such that, for some $\lambda_0 \in (0, 1/2)$,

$$X_t = (1 - 2\cos(2\pi\lambda_0)B + B^2)^{-\delta} W_t$$

Recursion for $\{g_k\}$ given by $g_{-1} = 0, g_0 = 1$ and for $k > 0$

$$g_k = \left(\frac{2}{k+1}\right)(\delta + k)\cos(2\pi\lambda_0) - \left(\frac{2\delta + k - 1}{k+1}\right) g_{k-1}$$

but no simple explicit form.

$\{g_k\}$ are coefficients of the *Gegenbauer* polynomials (see Gray, Zhang, Woodward (1989), Lapsa(1997)).

This procedure yields a process $\{X_t\}$ that has persistence **associated with the frequency** λ_0, and is **stationary**

- if $\delta < 1/2$ when $\lambda_0 \neq 0$, or
- if $\delta < 1/4$ when $\lambda_0 = 0$

SDF has relatively straightforward form

$$S(f) = \frac{1}{\left(2\left|\cos(2\pi f) - \cos(2\pi\lambda_0)\right|\right)^{2\delta}}$$

with

$$S(f) \to \frac{1}{\left(2\left|\sin(2\pi\lambda_0)\right|\right)^{2\delta}} \frac{1}{\left|2\pi f - 2\pi\lambda_0\right|^{2\delta}} \qquad f \to \lambda_0$$

ACV/ACF less straightforward

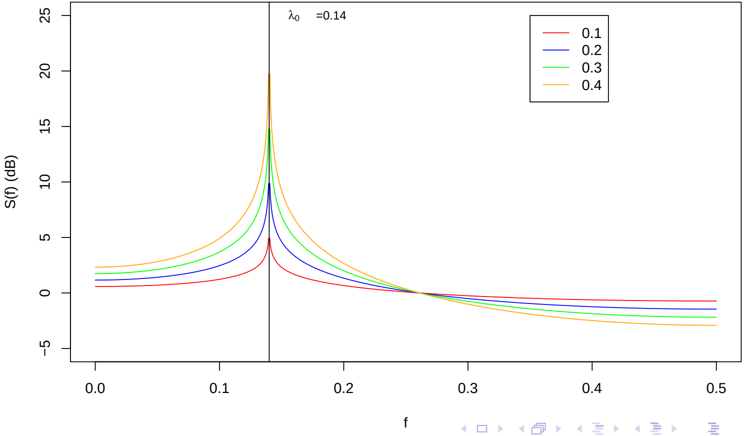
$$
\begin{aligned}
\sigma_X^2 \gamma_k \quad = \quad & \frac{\Gamma(1-2\delta)}{\sqrt{\pi}2^{1/2+2\delta}} \left\{\sin(2\pi\lambda_0)\right\}^{1/2-2\delta} \\
& \left[P_{k-1/2}^{2\delta-1/2}(\cos(2\pi\lambda_0)) + (-1)^k P_{k-1/2}^{2\delta-1/2}(-\cos(2\pi\lambda_0)) \right]
\end{aligned}
$$

where $P_\nu^\mu(x)$ is the associated Legendre function of the first kind.

A recursion formula for $P_\nu^\mu(x)$ gives the acvs to arbitrary lag.

$$
(\nu - \mu + 1)P_{\nu+1}^\mu(x) = (2\nu + 1)P_\nu^\mu(x) - (\nu + \mu)P_{\nu-1}^\mu(x)
$$

Gegenbauer Models Characteristic singularity (pole) in the spectrum at λ_0.



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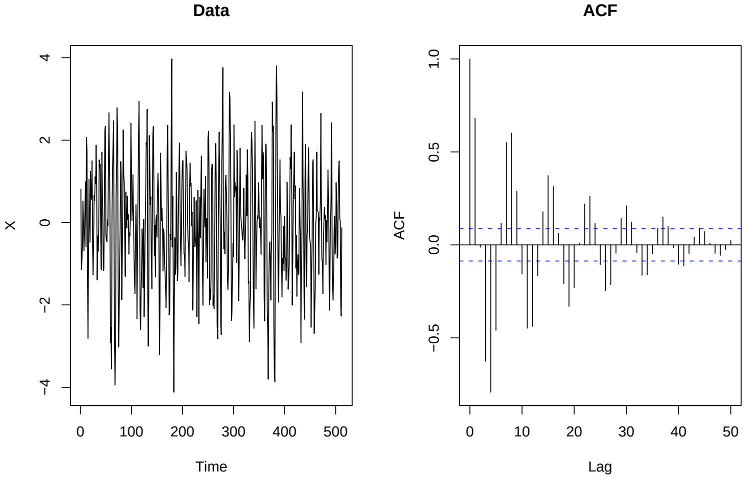
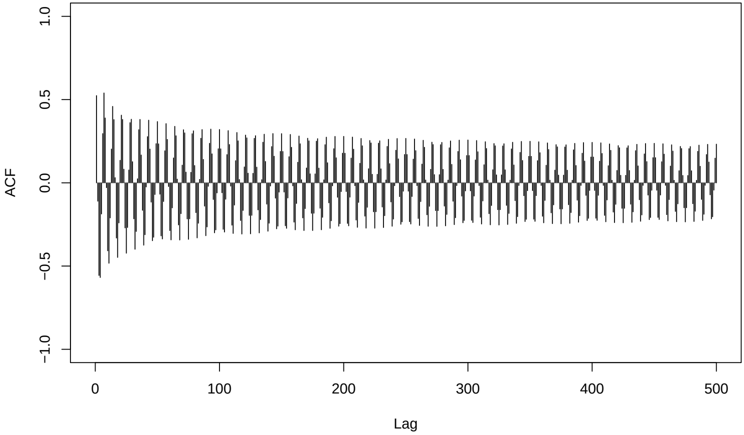
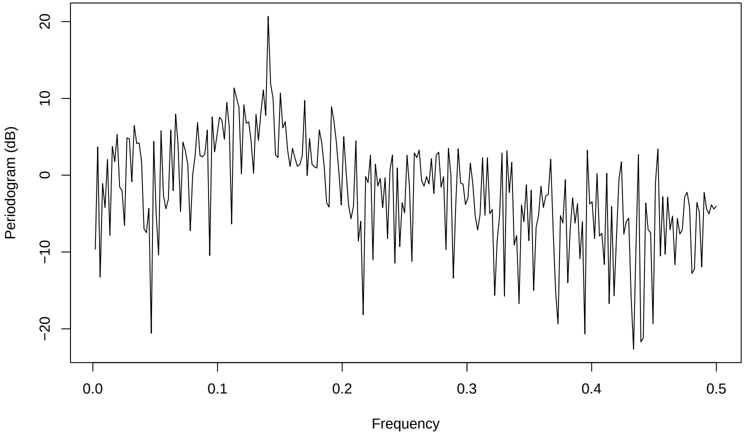
Example: $\lambda_0 = 0.14$, $\delta = 0.4$



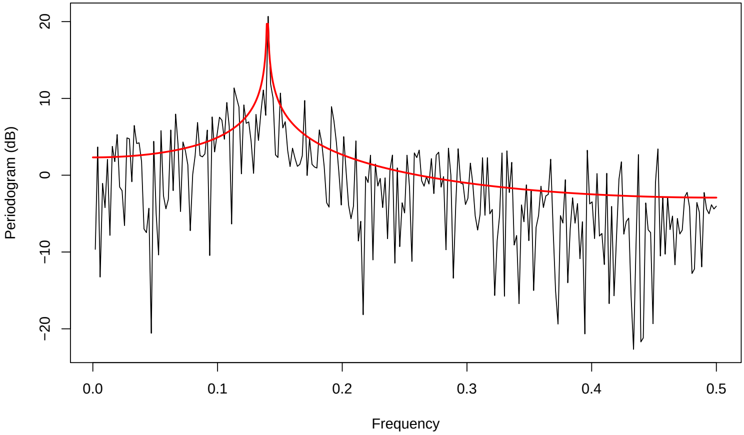
Theoretical ACF



Periodogram



Periodogram and SDF



Cointegration

Cointegration can be used to analyze co-movements in raw series (asset prices, exchange rates or yields).

The modelling strategy allows the detection of stable or stationary long-run relationships between non-stationary variables.

This allows the underlying relationships between the series to be discovered, and perhaps may allow forecasting.

The components of $K-$vector X_t are said to be *cointegrated* of order (d, b), denoted $X_t \sim CI(d, b)$ if

(a) the components of X_t are $I(d)$ (stationary after d-times differencing),

(b) there exists a linear combination of X_t, Z_t say, where

$$Z_t = \boldsymbol{\alpha}^{\mathsf{T}} X_t \qquad \boldsymbol{\alpha} \neq \mathbf{0}$$

such that $Z_t \sim I(b)$ for some $0 < b < d$. $\boldsymbol{\alpha}$ is termed the *cointegrating vector*.

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Two-Step Regression Strategy: Consider, for illustration, $Y_t, X_t = (X_{t1}, \ldots, X_{tK})^\mathsf{T} \sim I(1)$.

1. Form the regression model

$$Y_t = \alpha_1 X_{t1} + \ldots + \alpha_K X_{tK} + z_t$$

with cointegrating vector

$$\boldsymbol{\alpha} = (1, -\alpha_1, \ldots, \alpha_K)^\mathsf{T}$$

and estimate the parameters in the model (using OLS/maximum likelihood)

- Engle/Granger demonstrate that the resulting estimators are consistent.
- Under cointegration, residuals $\hat{z}_t$ should be $I(0)$; this can be tested using Dickey-Fuller procedures.

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2. If $\hat{z}_t \sim I(0)$ is acceptable (the unit root hypothesis is rejected) then an *Error Correction Model* (ECM) is specified of the form (for $K = 1$)

$$\begin{aligned} BY_t &= \psi_0 + \gamma_1 \hat{z}_{t-1} + \sum_{i=1}^{I} \psi_{1i} BX_{t-i} + \sum_{l=1}^{L} \psi_{2l} BY_{t-l} + \epsilon_{1t} \\ BX_t &= \xi_0 + \gamma_2 \hat{z}_{t-1} + \sum_{i=1}^{I} \xi_{1i} BY_{t-i} + \sum_{l=1}^{L} \xi_{2l} BY_{t-l} + \epsilon_{2t} \end{aligned}$$

Again, these models can be estimated using standard OLS/ML techniques.

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The ECM in the first equation states that changes in the series Y_t are explained by

- the error in the long-run equilibrium from previous time point (coefficient γ_1),
- lagged changes in the X_t series (coefficients ψ_1),
- their own history (coefficients ψ_2).

γ_1 determines the rate of re-adjustment; should have

$$\hat{\gamma}_1 < 0.$$

- Model orders l and L need to be selected; typically start with l, L large, and drop variables according to t-statistics, or using model selection criteria
- Can extend to systems of cointegrated variables

Vector Autoregression

Univariate methods of time series analysis can be extended to study parallel series.

Data may comprise

- stocks in a sector
- indices
- exchange rates

all which may exhibit evolution in time in some *dependent* fashion.

One extension is the *vector autoregression (VAR)*

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A Simple VAR structure Suppose that $\mathbf{Y}_t = (Y_{t1}, \ldots, Y_{td})^\mathsf{T}$ is a d-dimensional time series process. A suitable model for $\mathbf{Y}_t$ takes the form

$$\mathbf{Y}_t = \mathbf{X}_t\beta + \sum_{i=k}^{p} \mathbf{\Phi}_k \mathbf{Y}_{t-k} + \epsilon_t$$

where

- $\mathbf{X}_t\beta$ is a *deterministic* component
- Φ_k is a $d \times d$ matrix determining the dependence at lag $k = 1, 2, \ldots, p$.
- ϵ_t is zero mean, i.i.d. vector process with

$$E[\epsilon_t \epsilon_t^\mathsf{T}] = \Sigma$$

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- The resulting process is similar to the univariate $AR(p)$, and is denoted the $VAR(p)$ model.
- Usually
$$\epsilon_t \sim N(0, \Sigma)$$
is a suitable assumption. In this case, the resulting process is a multivariate Gaussian process.
- Estimation of the model can proceed using the usual likelihood methods.
- Restrictions on the model are required to preserve stationarity.

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Vector Cointegration/Error Correction Models

The ECM model of the previous section can be extended to cover the case of vectors of cointegrated variables.

A $d \times 1$ vector process $\mathbf{Y}_t$ is said to be cointegrated if at least one non-zero $d \times 1$ vector $\boldsymbol{\beta}_i$ exists such that

$$\beta_i^{\mathsf{T}}$$

is (trend) stationary.

If r such linearly independent vectors $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r$ exist, then $\mathbf{Y}_t$ is cointegrated with rank r, with

$$\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r)$$

the *cointegrating matrix*.

VECM:

$$\mathbf{BY}_t = \boldsymbol{\mu} + \mathbf{X}_t\beta + \boldsymbol{\Phi}\mathbf{Y}_{t-p} + \sum_{i=1}^{p-1} \boldsymbol{\Gamma}_k \mathbf{BY}_{t-k} + \epsilon_t$$

where

- $$\boldsymbol{\Gamma}_k = -(\mathbf{I} - \boldsymbol{\Phi}_1 - \ldots - \boldsymbol{\Phi}_i) \qquad i = 1, \ldots, p-1$$
- $$\boldsymbol{\Phi} = -(\mathbf{I} - \boldsymbol{\Phi}_1 - \ldots - \boldsymbol{\Phi}_p)$$

with all matrices $d \times d$.