Statistical Inference and Methods

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10th January 2006

Part III

Session 3: Time Series Analysis

Session 3: Time Series Analysis

- Exploratory Analysis
- Time Domain Models
- Frequency Domain modelling
- Inference and Estimation
- Non stationarity/Unit Roots

Session 3: Time Series Analysis

Time series analysis is a branch of **applied stochastic processes**. We start with an indexed family of **random variables**

$$\{X_t : \quad t \in T\}$$

where t is the index, here taken to be time (but it could be space). T is called the index set. We have a state space of values of X.

In addition X could be **univariate** or **multivariate**. We shall concentrate on discrete time. Samples are taken at equal intervals. We wish to use time series analysis to characterize time series and understand structure.

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State (possible values of X)	Time	Notation
Continuous	Continuous	$X(t)$
Continuous	Discrete	X_t
Discrete	Continuous	
Discrete	Discrete	

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Exploratory Analysis

We consider **lag** k scatter plots by plotting x_t versus x_{t+k}, but they are unwieldy. Suppose we make the assumption that a **linear** relationship holds approximately between x_{t+k} and x_t for all k, *i.e.*,

$$x_{t+k} = \alpha_k + \beta_k x_t + \varepsilon_{t+k}$$

where ε_{t+k} is an random error term.

The association between two variables $\{y_t\}$ and $\{z_t\}$ is the **Pearson product moment correlation coefficient**

$$\hat{\rho} = \frac{\sum(y_t - \bar{y})(z_t - \bar{z})}{\sqrt{\sum(y_t - \bar{y})^2 \sum(z_t - \bar{z})^2}}$$

where $\bar{y}$ and $\bar{z}$ are the sample means.

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Hence if $y_t = x_{t+k}$ and $z_t = x_t$ we are led to the lag k sample autocorrelation for a time series:

$$\hat{\rho}_k = \frac{\sum_{t=1}^{N-k} (x_{t+k} - \bar{x})(x_t - \bar{x})}{\sum_{t=1}^{N} (x_t - \bar{x})^2}$$

with $\hat{\rho}_0 = 1$.

The sequence $\{\hat{\rho}_k\}$ is called the **sample autocorrelation sequence** (sample acfs) for the time series.

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The series $x_1, \ldots, x_N$ can be regarded as a realization of the corresponding random variables $X_1, \ldots, X_N$, $\hat{\rho}_k$ is an estimate of a corresponding population quantity called the lag k theoretical autocorrelation, defined as

$$\rho_k = \frac{E\left[(X_t - \mu)(X_{t+k} - \mu)\right]}{\sigma^2}$$

where

$$\mu = E\left[X_t\right] \qquad \sigma^2 = E\left[(X_t - \mu)^2\right]$$

are the process mean and process variance.

Note that ρ_k, μ and σ^2 do not depend on t

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Denote the process by $\{X_t\}$. For fixed t, X_t is a random variable (r.v.), and hence there is an associated cumulative distribution function (cdf):

$$F_t(a) = P(X_t \leq a).$$

But we are interested in the relationships between the various r.v.s that form the process. For example, for any t_1 and $t_2 \in T$,

$$F_{t_1,t_2}(a_1, a_2) = P(X_{t_1} \leq a_1, X_{t_2} \leq a_2)$$

gives the bivariate cdf. More generally for any $t_1, t_2, \ldots, t_n \in T$,

$$F_{t_1,t_2,\ldots,t_n}(a_1, a_2, \ldots, a_n) = P(X_{t_1} \leq a_1, \ldots, X_{t_n} \leq a_n)$$

We consider the subclass of **stationary processes**.

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Stationarity

Strong stationarity $\{X_t\}$ is said to be strongly (strictly, completely) stationary if, for all $n \geq 1$, for any

$$t_1, t_2, \ldots, t_n \in T$$

and for any τ such that

$$t_1 + \tau, t_2 + \tau, \ldots, t_n + \tau \in T$$

are also contained in the index set, the joint cdf of $\{X_{t_1}, \ldots, X_{t_n}\}$ is the same as that of $\{X_{t_1+\tau}, \ldots, X_{t_n+\tau}\}$ i.e.,

$$F_{t_1,t_2,\ldots,t_n}(a_1, a_2, \ldots, a_n) = F_{t_1+\tau,t_2+\tau,\ldots,t_n+\tau}(a_1, a_2, \ldots, a_n).$$

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Second-order stationarity $\{X_t\}$ is said to be second-order (weakly) stationary if, for all $n \geq 1$, for any

$$t_1, t_2, \ldots, t_n \in T$$

and for any τ such that $t_1 + \tau, t_2 + \tau, \ldots, t_n + \tau \in T$ are also contained in the index set, all the joint moments of orders 1 and 2 of $\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\}$ exist and are finite.

Most importantly, these moments are identical to the corresponding joint moments of $\{X_{t_1+\tau}, X_{t_2+\tau}, \ldots, X_{t_n+\tau}\}$.

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Hence,

$$E[X_t] \equiv \mu \qquad Var[X_t] \equiv \sigma^2 \qquad (= E\left[X_t^2\right] - \mu^2),$$

are constants independent of t. If we let $\tau = -t_1$,

$$E[X_{t_1} X_{t_2}] = E[X_{t_1+\tau} X_{t_2+\tau}] = E[X_0 X_{t_2-t_1}],$$

and with $\tau = -t_2$,

$$E[X_{t_1} X_{t_2}] = E[X_{t_1+\tau} X_{t_2+\tau}] = E[X_{t_1-t_2} X_0].$$

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Hence, $E\{X_{t_1}X_{t_2}\}$ is a function of the absolute difference $|t_2 - t_1|$ only, similarly, for the **covariance** between X_{t_1} & X_{t_2}:

$$Cov\left[X_{t_1}, X_{t_2}\right] = E\left[(X_{t_1} - \mu)(X_{t_2} - \mu)\right] = E\left[X_{t_1}X_{t_2}\right] - \mu^2.$$

The **autocovariance sequence** (**acvs**), s_τ, is defined by

$$s_\tau \equiv Cov\left[X_t, X_{t+\tau}\right] = Cov\left[X_0, X_\tau\right].$$

- τ is called the lag.
- $s_0 = \sigma^2$ and $s_{-\tau} = s_\tau$, with $|s_\tau| \leq s_0$ for $\tau > 0$.
- The autocorrelation sequence (acfs) is given by

$$\rho_\tau = \frac{s_\tau}{s_0} = \frac{Cov\left[X_t, X_{t+\tau}\right]}{\sigma^2}.$$

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The sequence $\{s_\tau\}$ is positive semidefinite, i.e., for all $n \geq 1$, for any $t_1, t_2, \ldots, t_n$ contained in the index set, and for any set of nonzero real numbers $a_1, a_2, \ldots, a_n$

$$\sum_{j=1}^{n}\sum_{k=1}^{n} s_{t_j - t_k} a_j a_k \geq 0.$$

- Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)^{\mathsf{T}}$, $\mathbf{V} = (X_{t_1}, X_{t_2}, \ldots, X_{t_n})^{\mathsf{T}}$, and let Σ be the variance-covariance matrix of $\mathbf{V}$. Its j,k-th element is given by

$$s_{t_j - t_k} = E\left[(X_{t_j} - \mu)(X_{t_k} - \mu)\right]$$

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- Define the r.v.

$$w = \sum_{j=1}^{n} a_j X_{t_j} = \mathbf{a}^{\mathsf{T}} \mathbf{V}.$$

Then

$$\begin{aligned} 0 \leq Var\left[w\right] &= Var\left[\mathbf{a}^{\mathsf{T}}\mathbf{V}\right] = \mathbf{a}^{\mathsf{T}} Var\left[\mathbf{V}\right] \mathbf{a} = \mathbf{a}^{\mathsf{T}} \Sigma \mathbf{a} \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} s_{t_j - t_k} a_j a_k. \end{aligned}$$

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- The variance-covariance matrix of equispaced X's, $(X_1, X_2, \ldots, X_N)^{\mathsf{T}}$ has the form

$$\begin{bmatrix} s_0 & s_1 & \cdots & s_{N-2} & s_{N-1} \\ s_1 & s_0 & \cdots & s_{N-3} & s_{N-2} \\ \vdots & & \ddots & & \\ s_{N-2} & s_{N-3} & \cdots & s_0 & s_1 \\ s_{N-1} & s_{N-2} & \cdots & s_1 & s_0 \end{bmatrix}$$

  which is known as a symmetric Toeplitz matrix – all elements on a diagonal are the same.

- Note the above matrix has only N unique elements, $s_0, s_1, \ldots, s_{N-1}$.

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- A stochastic process $\{X_t\}$ is called Gaussian if, for all $n \geq 1$ and for any $t_1, t_2, \ldots, t_n$ contained in the index set, the joint cdf of $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ is multivariate Gaussian.
- 2nd-order stationary Gaussian $\Rightarrow$ complete stationarity
- follows as the multivariate Normal distribution is completely characterized by 1st and 2nd moments
- not true in general.
- Complete stationarity $\Rightarrow$ 2nd-order stationary in general.

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White noise process

Also known as a purely random process. Let $\{X_t\}$ be a sequence of uncorrelated r.v.s such that

$$E[X_t] = \mu \qquad Var[X_t] = \sigma^2 \qquad \forall t$$

and

$$s_\tau = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases} \quad \text{or} \quad \rho_\tau = \begin{cases} 1 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases}$$

forms a basic building block in time series analysis. Very different realizations of white noise can be obtained for different distributions of $\{X_t\}$.

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q-th order moving average process MA(q)
X_t can be expressed in the form

$$X_t = \mu - \theta_{0,q}\epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q} = \mu - \sum_{j=0}^{q} \theta_{j,q}\epsilon_{t-j},$$

where μ and $\theta_{j,q}$'s are constants ($\theta_{0,q} \equiv -1, \theta_{q,q} \neq 0$), and $\{\epsilon_t\}$ is a zero-mean white noise process with variance σ^2_ϵ.
We assume $E[X_t] = \mu = 0$. Then

$$Cov[X_t, X_{t+\tau}] = E\{X_t X_{t+\tau}\}$$

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Recall: $Cov(X, Y) = E\{(X - E\{X\})(Y - E\{Y\})\}$. Since $E\{\epsilon_t \epsilon_{t+\tau}\} = 0 \ \forall \tau \neq 0$ we have for $\tau \geq 0$.

$$\begin{aligned} Cov\left[X_t, X_{t+\tau}\right] &= \sum_{j=0}^{q}\sum_{k=0}^{q} \theta_{j,q}\theta_{k,q} E\{\epsilon_{t-j}\epsilon_{t+\tau-k}\} \\ &= \sigma_\epsilon^2 \sum_{j=0}^{q-\tau} \theta_{j,q}\theta_{j+\tau,q} \qquad (k = j + \tau) \\ &\equiv s_\tau, \end{aligned}$$

which does not depend on t.

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Since $s_\tau = s_{-\tau}$, $\{X_t\}$ is a stationary process with acvs given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q}\theta_{j+|\tau|,q} & |\tau| \leq q \\ 0 & |\tau| > q \end{cases}$$

No restrictions were placed on the $\theta_{j,q}$'s to ensure stationarity.

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Example: $X_t = \epsilon_t - \theta_{1,1}\epsilon_{t-1}$ MA(1)

acvs:

$$s_\tau = \sigma_\epsilon^2 \sum_{j=0}^{1-|\tau|} \theta_{j,1}\theta_{j+|\tau|,1} \quad |\tau| \leq 1,$$

so,

$$s_0 = \sigma_\epsilon^2(\theta_{0,1}\theta_{0,1} + \theta_{1,1}\theta_{1,1}) = \sigma_\epsilon^2(1 + \theta_{1,1}^2);$$

and,

$$s_1 = \sigma_\epsilon^2\theta_{0,1}\theta_{1,1} = -\sigma_\epsilon^2\theta_{1,1}.$$

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acfs:

$$\rho_\tau = \frac{s_\tau}{s_0} : \rho_0 = 1.0 \qquad \rho_1 = \frac{-\theta_{1,1}}{1+\theta_{1,1}^2}$$

For $\theta_{1,1} = 1.0, \sigma_\epsilon^2 = 1.0$, we have,

$$s_0 = 2.0, s_1 = -1.0, s_2, s_3, \ldots = 0.0,$$

giving,

$$\rho_0 = 1.0, \rho_1 = -0.5, \rho_2, \rho_3, \ldots = 0.0.$$

For $\theta_{1,1} = -1.0, \sigma_\epsilon^2 = 1.0$, we have,

$$s_0 = 2.0, s_1 = 1.0, s_2, s_3, \ldots = 0.0,$$

giving,

$$\rho_0 = 1.0, \rho_1 = 0.5, \rho_2, \rho_3, \ldots = 0.0.$$

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Note: if we replace $\theta_{1,1}$ by $\theta_{1,1}^{-1}$ the model becomes

$$X_t = \epsilon_t - \frac{1}{\theta_{1,1}}\epsilon_{t-1}$$

and the autocorrelation becomes

$$\rho_1 = \frac{-\frac{1}{\theta_{1,1}}}{1+\left(\frac{1}{\theta_{1,1}}\right)^2} = \frac{-\theta_{1,1}}{\theta_{1,1}^2+1},$$

i.e., is unchanged. Thus we cannot identify the MA(1) process uniquely from the autocorrelation.

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p-th order autoregressive process AR(p)
$\{X_t\}$ is expressed in the form

$$X_t = \phi_{1,p} X_{t-1} + \phi_{2,p} X_{t-2} + \ldots + \phi_{p,p} X_{t-p} + \epsilon_t,$$

where $\phi_{1,p}, \phi_{2,p}, \ldots, \phi_{p,p}$ are constants ($\phi_{p,p} \neq 0$) and $\{\epsilon_t\}$ is a zero mean white noise process with variance σ^2_ϵ.

In contrast to the parameters of an MA(q) process, the $\{\phi_{k,p}\}$ **must satisfy certain conditions** for $\{X_t\}$ to be a stationary process – not all AR(p) processes are stationary.

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Example

$$\begin{aligned} X_t &= \phi_{1,1} X_{t-1} + \epsilon_t \\ &= \phi_{1,1}\{\phi_{1,1} X_{t-2} + \epsilon_{t-1}\} + \epsilon_t \\ &= \phi_{1,1}^2 X_{t-2} + \phi_{1,1}\epsilon_{t-1} + \epsilon_t \\ &\vdots \\ &= \sum_{k=0}^{\infty} \phi_{1,1}^k \epsilon_{t-k} \qquad \text{(initial condition } X_{-N} = 0; \text{ let } N \to \infty \end{aligned}$$

$$Var\left[X_t\right] = Var\left[\sum_{k=0}^{\infty} \phi_{1,1}^k \epsilon_{t-k}\right] = \sum_{k=0}^{\infty} Var\{\phi_{1,1}^k \epsilon_{t-k}\} = \sigma_\epsilon^2 \sum_{k=0}^{\infty} \phi_{1,1}^{2k}.$$

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For $Var[X_t] < \infty$ we must have $|\phi_{1,1}| < 1$, in which case

$$Var[X_t] = \frac{\sigma_\epsilon^2}{1 - \phi_{1,1}^2}.$$

To find the form of the acvs, we notice that for $\tau > 0$, $X_{t-\tau}$ is a linear function of $\epsilon_{t-\tau}, \epsilon_{t-\tau-1}, \ldots$ and is therefore uncorrelated with ϵ_t. Hence

$$E[\epsilon_t X_{t-\tau}] = 0.$$

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Assuming stationarity and multiplying the defining equation (1) by $X_{t-\tau}$:

$$X_t X_{t-\tau} = \phi_{1,1} X_t X_{t-\tau} + \epsilon_t X_{t-\tau}$$

$$\Longrightarrow E\left[X_t X_{t-\tau}\right] = \phi_{1,1} E\left[X_{t-1} X_{t-\tau}\right]$$

so that

$$s_\tau = \phi_{1,1} s_{\tau-1} = \phi_{1,1}^2 s_{\tau-2} = \ldots = \phi_{1,1}^\tau s_0 \qquad \Rightarrow \rho_\tau = \frac{s_\tau}{s_0} = \phi_{1,1}^\tau$$

However ρ_τ is an even function of τ, so

$$\rho_\tau = \phi_{1,1}^{|\tau|} \qquad \tau = 0, \pm 1, \pm 2, \ldots.$$

giving exponential decay

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(p, q)'th order autoregressive-moving average process ARMA(p, q)

Here $\{X_t\}$ is expressed as

$$X_t = \phi_{1,p} X_{t-1} + \ldots + \phi_{p,p} X_{t-p} + \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q},$$

where the $\phi_{j,p}$'s and the $\theta_{j,q}$'s are all constants $(\phi_{p,p} \neq 0; \theta_{q,q} \neq 0)$ and again $\{\epsilon_t\}$ is a zero mean white noise process with variance σ^2_ϵ.

The ARMA class is important as many data sets may be approximated in a more parsimonious way (meaning fewer parameters are needed) by a mixed ARMA model than by a pure AR or MA process.

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The General Linear Process

Consider a process of the form

$$X_t = \sum_{k=-\infty}^{\infty} g_k \epsilon_{t-k},$$

where $\{\epsilon_t\}$ is a purely random process, with

$$\sum_{k=-\infty}^{\infty} g_k^2 < \infty.$$

This condition ensures that $\{X_t\}$ has finite variance. Now $|\rho_t| \leq 1$, so, also,

$$|s_\tau| = |Cov\,[X_t, X_{t-\tau}]| \leq \sigma_X^2 = \sigma_\epsilon^2 \sum_k g_k^2 < \infty.$$

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If $g_{-1}, g_{-2}, \ldots = 0$, then we obtain what is called the *General Linear Process*

$$X_t = \sum_{k=0}^{\infty} g_k \epsilon_{t-k},$$

where X_t depends only on past and present values $\epsilon_t, \epsilon_{t-2}, \epsilon_{t-2}, \ldots$ of the purely random process. Consider the function

$$G(z) = \sum_{k=0}^{\infty} g_k z^k,$$

"z-polynomial" where $z = e^{-i\omega}$. Note $X_t = G(B)\epsilon_t$.

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Then write

$$G(z) = \frac{G_1(z)}{G_2(z)}$$

Call the zeros of $G_2(z)$ (the "poles" of $G(z)$) in the **complex plane** $z_1, z_2, \ldots, z_p$, where the zeros are ordered so that $z_1, \ldots, z_k$ are inside and $z_{k+1}, \ldots, z_p$ are outside the unit circle $|z| = 1$.

Then, if all the roots of $G_2(z)$ are outside the unit circle (i.e. all the poles of $G(z)$ are outside the unit circle) only past and present values of $\{\epsilon_t\}$ are involved and the General Linear Process exists.

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Another way of stating this is that

$$G(z) < \infty \qquad |z| \leq 1$$

i.e., $G(z)$ is analytic inside and on the unit circle. Thus

- all the poles of $G(z)$ lie outside the unit circle
- all the roots of $G^{-1}(z) = 0$ lie outside the unit circle

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Consider the MA(q) model

$$X_t = \Theta(B)\epsilon_t \qquad \Longrightarrow \qquad \Theta^{-1}(B)X_t = \epsilon_t$$

and in general, the expansion of $\Theta^{-1}(B)$ is a polynomial of infinite order. Similarly, consider the AR(p) model

$$\Phi(B)X_t = \epsilon_t \qquad \Longrightarrow \qquad X_t = \Phi^{-1}(B)\epsilon_t.$$

Hence

$$\begin{array}{lcl} \text{MA (finite order)} & \equiv & \text{AR (infinite order)} \\ \text{AR (finite order)} & \equiv & \text{MA (infinite order)} \end{array}$$

provided the infinite order expansions exist

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Invertibility

Consider inverting the general linear process into autoregressive form

$$\begin{aligned} X_t &= \sum_{k=0}^{\infty} g_k \epsilon_{t-k} = \sum_{k=0}^{\infty} g_k B^k \epsilon_t \\ &= G(B)\epsilon_t \end{aligned}$$

so that

$$G^{-1}(B)X_t = \epsilon_t$$

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The expansion of $G^{-1}(B)$ in powers of B gives the required autoregressive form **provided** $G^{-1}(B)$ **admits a power series expansion**

$$G^{-1}(z) = \sum_{k=0}^{\infty} h_k z^k$$

i.e. if $G^{-1}(z)$ is analytic, $|z| \leq 1$. Thus the model is invertible if all the poles of $G^{-1}(z)$ are outside the unit circle.

$$G^{-1}(z) < \infty, \quad |z| \leq 1.$$

For the MA(q) process, $G(z) = \Theta(z)$, and so the invertibility condition is that $\Theta(z)$ has no roots inside or on the unit circle; i.e. all the roots of $\Theta(z)$ lie outside the unit circle.

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Stationarity of ARMA processes

For the AR(p) process

$$\Phi(B)X_t = \epsilon_t$$

so that

$$X_t = \Phi^{-1}(B)\epsilon_t = G(B)\epsilon_t,$$

so that $G(z) = \Phi^{-1}(z)$. Hence the requirement for stationarity is that all the roots of $G^{-1}(z) = \Phi(z)$ must lie outside the unit circle.

For the MA(q) process

$$X_t = \Theta(B)\epsilon_t = G(B)\epsilon_t$$

and since $G(B) = \Theta(B)$ is a polynomial of finite order $G(z) < \infty$, $|z| \leq 1$, automatically.

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Example:

$$X_t = 1.3X_{t-1} - 0.4X_{t-2} + \epsilon_t - 1.5\epsilon_{t-1}.$$

Writing in B notation:

$$(1 - 1.3B + 0.4B^2)X_t = (1 - 1.5B)\epsilon_t$$

we have

$$\Phi(z) = 1 - 1.3z + 0.4z^2$$

with roots $z = 2$ and $5/4$, so the roots of $\Phi(z) = 0$ both lie outside the unit circle, and the model is stationary, and

$$\Theta(z) = 1 - 1.5z,$$

so the root of $\Theta(z) = 0$ is given by $z = 2/3$ which lies inside the unit circle and the model is not invertible.

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Directionality and Reversibility

Consider again the general linear model

$$X_t = \sum_{k=0}^{\infty} g_k \epsilon_{t-k} = \sum_{k=0}^{\infty} g_k B^k \epsilon_t = G(B)\epsilon_t$$

The reversed form is clearly,

$$X_t = \sum_{k=0}^{\infty} g_k \epsilon_{t+k} = \sum_{k=0}^{\infty} g_k B^{-k} \epsilon_t = G\left(\frac{1}{B}\right)\epsilon_t,$$

with some stationarity condition.

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Now consider the ARMA(p, q) model given by

$$\Phi(B)X_t = \Theta(B)\epsilon_t,$$

where,

$$\begin{aligned}\Phi(B) &= 1 - \phi_{1,p}B - \phi_{2,p}B^2 - \ldots - \phi_{p,p}B^p \\ \Theta(B) &= 1 - \theta_{1,q}B - \theta_{2,q}B^2 - \ldots - \theta_{q,q}B^q\end{aligned}$$

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The reversed form of the ARMA(p, q) model is,

$$\Phi\left(\frac{1}{B}\right) X_t = \Theta\left(\frac{1}{B}\right)\epsilon_t \Longrightarrow \Phi^R(B)X_t = B^{p-q}\Theta^R\epsilon_t$$

where,

$$\begin{aligned}\Phi^R(B) &= B^p - \phi_{1,p}B^{p-1} - \phi_{2,p}B^{p-2} - \ldots - \phi_{p,p}\\ \Theta^R(B) &= B^q - \theta_{1,q}B^{q-1} - \theta_{2,q}B^{q-2} - \ldots - \theta_{q,q}\end{aligned}$$

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For example, for the ARMA(1,1) model,

$$(1 - \phi_{1,1})X_t = (1 - \theta_{1,1})\epsilon_t,$$

reversed form is

$$(B - \phi_{1,1})X_t = (B - \theta_{1,1})\epsilon_t$$

Now $\Phi(z) = 1 - \phi_{1,1}z$, and a root is the solution of $1 - \phi_{1,1}z = 0$, i.e.,

$$|z| = \left|\frac{1}{\phi_{1,1}}\right| > 1 \Rightarrow |\phi_{1,1}| < 1.$$

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But, $\Phi^R(z) = z - \phi_{1,1}$, and so a root is the solution of $z - \phi_{1,1} = 0$, i.e., $z = \phi_{1,1}$. But, since for stationarity $|\phi_{1,1}| < 1$ we have

$$|z| = |\phi_{1,1}| < 1,$$

so the root of $\Phi^R(z)$ is inside the unit circle.

Hence the standard assumption for stationarity (roots outside the unit circle) has within it an assumption of directionality. [N.B. only if the roots of $\Phi(z)$ are on the unit circle is model ALWAYS non-stationary].

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Spectral Representations

Spectral analysis is a study of the frequency domain characteristics of a process, and describes the contribution of each frequency to the variance of the process. Let us define a complex "jump" process $\{Z(f)\}$ on the interval $[0, 1/2]$, such that

$$dZ(f) \equiv \begin{cases} Z(f+df) - Z(f), & 0 \leq f < 1/2; \\ 0, & f = 1/2; \\ dZ^*(-f), & -1/2 \leq f < 0, \end{cases}$$

where df is a small positive increment. If the intervals $[f, f+df]$ and $[f', f'+df']$ are non-intersecting subintervals of $[-1/2, 1/2]$, then the r.v.'s $dZ(f)$ and $dZ(f')$ are uncorrelated.

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We say that the process has **orthogonal increments**, and the process itself is called an **orthogonal process** – this orthogonality results is very important.

Let $\{X_t\}$ be a real-valued discrete time stationary process, with zero mean, the **spectral representation theorem** states that there exists such an orthogonal process $\{Z(f)\}$, defined on $(-1/2, 1/2]$, such that

$$X_t = \int_{-1/2}^{1/2} e^{i2\pi ft}\, dZ(f)$$

for all integers t.

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The process $\{Z(f)\}$ has the following properties:

- $E\{dZ(f)\} = 0 \ \forall \ |f| \leq 1/2.$
- $E\{|dZ(f)|^2\} \equiv dS^{(I)}(f)$ say $\forall \ |f| \leq 1/2$, where $dS^{(I)}(f)$ is called the integrated spectrum of $\{X_t\}$, and
- for any two distinct frequencies f and $f' \in (-1/2, 1/2]$

$$Cov\{dZ(f'), dZ(f)\} = E\{dZ^*(f')dZ(f)\} = 0.$$

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The spectral representation

$$X_t = \int_{-1/2}^{1/2} e^{i2\pi ft}\, dZ(f) = \int_{-1/2}^{1/2} e^{i2\pi ft}\, |dZ(f)| e^{i \arg\{dZ(f)\}},$$

means that we can represent any discrete stationary process as an "infinite" sum of complex exponentials at frequencies f with associated random amplitudes $|dZ(f)|$ and random phases $\arg\{dZ(f)\}$.

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The orthogonal increments property can be used to define the relationship between the autocovariance sequence $\{s_\tau\}$ and the integrated spectrum $S^I(f)$:

$$s_\tau = E[X_t X_{t+\tau}] = E[X_t^* X_{t+\tau}]$$

$$= E\left[\int_{-1/2}^{1/2} e^{-i2\pi f' t}\, dZ^*(f') \int_{-1/2}^{1/2} e^{i2\pi f(t+\tau)}\, dZ(f)\right]$$

$$= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{i2\pi(f-f')t} e^{i2\pi f\tau} E\{dZ^*(f')dZ(f)\}.$$

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Because of the orthogonal increments property,

$$E\{dZ^*(f')dZ(f)\} = dS^{(I)}(f) \qquad f = f'$$

and zero otherwise, so

$$s_\tau = \int_{-1/2}^{1/2} e^{i2\pi f\tau}\, dS^{(I)}(f),$$

which shows that the integrated spectrum determines the acvs for a stationary process. If $S^{(I)}(f)$ is differentiable with derivative $S(f)$ (the *spectral density function* (sdf)), we have

$$E\{|dZ(f)|^2\} = dS^{(I)}(f) = S(f)\, df.$$

Hence

$$s_\tau = \int_{-1/2}^{1/2} e^{i2\pi ft} S(f)\, df.$$

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But a square summable deterministic sequence $\{g_t\}$ say has the **Fourier representation**

$$g_t = \int_{-1/2}^{1/2} G(f) e^{i2\pi ft}\, df \qquad \text{where} \qquad G(f) = \sum_{t=-\infty}^{\infty} g_t e^{-i2\pi ft},$$

If we assume that $S(f)$ is square integrable, then $S(f)$ is the **Fourier transform** of $\{s_\tau\}$,

$$S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f\tau}.$$

Hence,

$$\{s_\tau\} \longleftrightarrow S(f),$$

i.e., $\{s_\tau\}$ and $S(f)$ are a FT. pair.

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$S(\cdot)$ has the following interpretation: $S(f)\,df$ is the average contribution (over all realizations) to the power from components with frequencies in a small interval about f. The power – or variance – is

$$\int_{-1/2}^{1/2} S(f)\,df.$$

Hence, $S(f)$ is often called the *power spectral density function* or just *power spectrum*.

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Properties :

- $S^{(I)}(f) = \int_{-1/2}^{f} S(f')\, df'$.
- $0 \leq S^{(I)}(f) \leq \sigma^2$ where $\sigma^2 = Var\left[X_t\right]$; $\quad S(f) \geq 0$.
- $S^{(I)}(-1/2) = 0$; $\quad S^{(I)}(1/2) = \sigma^2$; $\quad \int_{-1/2}^{1/2} S(f)\, df = \sigma^2$.
- $f < f' \Rightarrow S^{(I)}(f) \leq S^{(I)}(f')$; $\quad S(-f) = S(f)$.

Except, basically, for the scaling factor σ^2, $S^{(I)}(f)$ has all the properties of a probability distribution function, and hence is sometimes called a **spectral distribution function**.

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The integrated spectrum, $S^{(I)}(f)$ can be decomposed as

$$S^{(I)}(f) = S_1^{(I)}(f) + S_2^{(I)}(f)$$

where the $S_j^{(I)}(f)$'s are nonnegative, nondecreasing functions with $S_j^{(I)}(-1/2) = 0$ and are of the following types:

- $S_1^{(I)}(\cdot)$ has its derivative $S(\cdot)$ for all f, and

$$S^{(I)}(f) = \int_{-1/2}^{f} S(f')df'.$$

- $S_2^{(I)}(\cdot)$ is a step function with jumps of size $\{p_l\}: \ l = 1, 2, \ldots\}$ at the points $\{f_l: \ l = 1, 2, \ldots\}$.

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(a) If $S_1^{(I)}(f) \geq 0; S_2^{(I)}(f) = 0$, $\{X_t\}$ has a *purely continuous* spectrum and $S(f)$ is absolutely integrable, with

$$\int_{-1/2}^{1/2} S(f)\cos(2\pi f\tau)\,df \quad \text{and} \quad \int_{-1/2}^{1/2} S(f)\sin(2\pi f\tau) \longrightarrow 0,$$

as $\tau \longrightarrow \infty$. But,

$$\begin{aligned} s_\tau &= \int_{-1/2}^{1/2} e^{i2\pi f\tau} S(f)\,df \\ &= \int_{-1/2}^{1/2} S(f)\cos(2\pi f\tau)\,df + i\int_{-1/2}^{1/2} S(f)\sin(2\pi f\tau)\,df \end{aligned}$$

so that $s_\tau \longrightarrow 0$ as $|\tau| \longrightarrow \infty$. In other words, the acvs diminishes to zero (called "mixing condition").

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(b) If $S_1^{(I)}(f) = 0; S_2^{(I)}(f) \geq 0$, the integrated spectrum consists entirely of a step function, and the $\{X_t\}$ is said to have a *purely discrete spectrum* or a *line spectrum* .

The acvs for a process with a line spectrum never damps down to 0.

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White noise spectrum

Recall that a white noise process $\{\epsilon_t\}$ has acvs:

$$s_\tau = \begin{cases} \sigma_\epsilon^2 & \tau = 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the spectrum of a white noise process is given by:

$$S_\epsilon(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f\tau} = s_0 = \sigma_\epsilon^2.$$

i.e., white noise has a constant spectrum.

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The sdf and acvs contain the same amount of information in that if we know one of them, we can calculate the other. However, they are often not equally informative.

- The sdf usually proves to be the more sensitive and interpretable diagnostic or exploratory tool.
- The sdf is able to distinguish between the processes while the acvs's are not noticeably different.
- dB $= 10 \log_{10}(\text{power})$ scale often used.

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Sampling and Aliasing

So far we have only looked at discrete time series $\{X_t\}$. However, such a process is usually obtained by sampling a continuous time process at equal intervals Δt, i.e., for a sampling interval $\Delta t > 0$ and an arbitrary time offset t_0, we can define a discrete time process through

$$X_t \equiv X(t_0 + t\Delta t), \qquad t = 0, \pm 1, \pm 2, \ldots.$$

If $\{X(t)\}$ is a stationary process with, say, sdf $S_{X(t)}(\cdot)$ and acvf $s(\tau)$, then $\{X_t\}$ is also a stationary process with, say, sdf $S_{X_t}(\cdot)$ and acvs $\{s_\tau\}$.

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It can be shown that when $S^{(I)}_{X(t)}$ is differentiable:

$$S_{X_t}(f) = \sum_{k=-\infty}^{\infty} S_{X(t)}\left(f + \frac{k}{\Delta t}\right) \quad \text{for} \quad |f| \leq \frac{1}{2\Delta t}.$$

Thus, the discrete time sdf at f is the sum of the continuous time sdf at frequencies $f \pm \frac{k}{\Delta t}$, $k = 0, 1, 2, \ldots$.

The frequency $1/(2\Delta t)$ is called the *Nyquist frequency*; previously we have taken $\Delta t = 1$, so that the frequency range was $|f| \leq \frac{1}{2}$.

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If $S_{X(t)}$ is essentially zero for $|f| > 1/(2\Delta t)$ we can expect good correspondence between $S_{X_t}(f)$ and $S_{X(t)}(f)$ for $|f| \leq 1/(2\Delta t)$ (since

$$S_{X(t)}(f \pm k/(2\Delta t)) \approx 0$$

for $k = 1, 2, \ldots$).

If $S_{X(t)}$ is large for some $|f| > 1/(2\Delta t)$, the correspondence can be quite poor, and an estimate of S_{X_t} will not tell us much about $S_{X(t)}$.

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Estimation and Forecasting

Ergodic Property Methods we shall look at for estimating quantities such as the autocovariance function will use observations from a single realization.

Such methods are based on the strategy of replacing ensemble averages by their corresponding time averages.

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Sample mean:

Given a time series $X_1, X_2, \ldots, X_N$, let

$$\bar{X} = \frac{1}{N}\sum X_t. \quad \left(\text{assume} \sum_{\tau=-\infty}^{\infty} |s_\tau| < \infty\right).$$

Then,

$$E\{\bar{X}\} = \frac{1}{N}\sum_{t=1}^{n} E\left[X_t\right] = \frac{1}{N}.N\mu = \mu$$

so $\bar{X}$ is an unbiased estimator of μ. Hence, $\bar{X}$ converges to μ in mean square if

$$\lim_{N\to\infty} Var\{\bar{X}\} = 0.$$

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$$
\begin{aligned}
Var\{\bar{X}\} &= E\{(\bar{X}-\mu)^2\} = E\left\{\left(\frac{1}{N}\sum_{i=1}^{N}(X_i-\mu)\right)^2\right\} \\
&= \frac{1}{N^2}\sum_{t=1}^{N}\sum_{u=1}^{N}E\{(X_t-\mu)(X_u-\mu)\} = \frac{1}{N^2}\sum_{t=1}^{N}\sum_{u=1}^{N}s_{u-t} \\
&= \frac{1}{N^2}\sum_{\tau=-(N-1)}^{N-1}\sum_{k=1}^{N-|\tau|}s_\tau \\
&= \frac{1}{N^2}\sum_{\tau=-(N-1)}^{N-1}(N-|\tau|)s_\tau = \frac{1}{N}\sum_{\tau=-(N-1)}^{N-1}\left(1-\frac{|\tau|}{N}\right)s_\tau
\end{aligned}
$$

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If

$$\sum_{\tau=-(N-1)}^{N-1} s_\tau$$

converges to a limit as $N \longrightarrow \infty$, then

$$\text{it must since} \quad \left| \sum_{\tau=-(N-1)}^{N-1} s_\tau \right| \leq \sum_{\tau=-(N-1)}^{N-1} |s_\tau| < \infty \ \ \forall\, N,$$

then $\sum_{\tau=-(N-1)}^{N-1} \left(1 - \frac{|\tau|}{N}\right) s_\tau$ converges to the same limit.

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We can thus conclude that,

$$\begin{aligned}\lim_{N\to\infty} NVar\{\bar{X}\} &= \lim_{N\to\infty} \sum_{\tau=-(N-1)}^{N-1} \left(1 - \frac{|\tau|}{N}\right) s_\tau \\ &= \lim_{N\to\infty} \sum_{\tau=-(N-1)}^{N-1} s_\tau = \sum_{\tau=-\infty}^{\infty} s_\tau .\end{aligned}$$

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The assumption of absolute summability of $\{s_\tau\}$ implies that $\{X_t\}$ has a purely continuous spectrum with sdf

$$S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f\tau}, \qquad \text{so that } S(0) = \sum_{\tau=-\infty}^{\infty} s_\tau.$$

Thus

$$\lim_{N\to\infty} NVar\{\bar{X}\} = S(0) \qquad \therefore \qquad Var\{\bar{X}\} \approx \frac{S(0)}{N} \quad \text{for large } N.$$

and therefore, $Var\{\bar{X}\} \to 0$. Note that the convergence of $\bar{X}$ depends only on the spectrum at $S(0)$, i.e. at $f = 0$.

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Autocovariance Sequence: Now,

$$s_\tau = E\{(X_t - \mu)(X_{t+\tau} - \mu)\}$$

so that a natural estimator for the acvs is

$$\hat{s}_\tau^{(u)} = \frac{1}{N - |\tau|} \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X}) \quad \tau = 0, \pm 1, \ldots, \pm(N-1).$$

Note $\hat{s}_{-\tau}^{(u)} = \hat{s}_\tau^{(u)}$ as it should.

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If we replace $\bar{X}$ by μ:

$$\begin{aligned} E\{\hat{s}_\tau^{(u)}\} &= \frac{1}{N-|\tau|}\sum_{t=1}^{N-|\tau|} E\{(X_t-\mu)(X_{t+|\tau|}-\mu)\} \\ &= \frac{1}{N-|\tau|}\sum_{t=1}^{N-|\tau|} s_\tau = s_\tau, \quad \tau = 0, \pm 1, \ldots, \pm(N-1). \end{aligned}$$

Thus, $\hat{s}_\tau^{(u)}$ is an unbiased estimator of s_τ when μ is known. (Hence the (u) – for unbiased). Most texts refer to $\hat{s}_\tau^{(u)}$ as unbiased – however, if μ is estimated by $\bar{X}$, $\hat{s}_\tau^{(u)}$ is typically a biased estimator of s_τ.

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A second estimator of s_τ is typically preferred:

$$\hat{s}_\tau^{(p)} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X}) \quad \tau = 0, \pm 1, \ldots, \pm(N-1).$$

With $\bar{X}$ replaced by μ:

$$E\{\hat{s}_\tau^{(p)}\} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} s_\tau = \left(1 - \frac{|\tau|}{N}\right) s_\tau,$$

so that $\hat{s}_\tau^{(p)}$ is a biased estimator, and the magnitude of its bias increases as $|\tau|$ increases. Most texts refer to $\hat{s}_\tau^{(p)}$ as biased.

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Why should we prefer the "biased" estimator $\hat{s}_\tau^{(p)}$ to the "unbiased" estimator $\hat{s}_\tau^{(u)}$?

1. For many stationary processes of practical interest

$$\text{mse}\{\hat{s}_\tau^{(p)}\} < \text{mse}\{\hat{s}_\tau^{(u)}\},$$

where

$$\begin{aligned}
\text{mse}\{\hat{s}_\tau\} &= E\{(\hat{s}_\tau - s_\tau)^2\} \\
&= E\{\hat{s}_\tau^2\} - 2s_\tau E\{\hat{s}_\tau\} + s_\tau^2 \\
&= (E\{\hat{s}_\tau^2\} - E^2\{\hat{s}_\tau\}) + E^2\{\hat{s}_\tau\} - 2s_\tau E\{\hat{s}_\tau\} + s_\tau^2 \\
&= Var\{\hat{s}_\tau\} + (s_\tau - E\{\hat{s}_\tau\})^2 \\
&= \text{variance} + (\text{bias})^2
\end{aligned}$$

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2 If $\{X_t\}$ has a purely continuous spectrum we know that $s_\tau \longrightarrow 0$ as $|\tau| \longrightarrow \infty$. It therefore makes sense to choose an estimator that decreases nicely as $|\tau| \longrightarrow N-1$ (i.e. choose $\hat{s}_\tau^{(p)}$).

3 We know that the acvs must be positive semidefinite, the sequence $\{\hat{s}_\tau^{(p)}\}$ has this property, whereas the sequence $\{\hat{s}_\tau^{(u)}\}$ may not.

Session 3: Time Series Analysis

The Periodogram

Suppose

$$S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f\tau} \quad |f| \leq \frac{1}{2},$$

is purely continuous. We can use the (biased) estimator of s_τ:

$$\hat{s}_\tau^{(p)} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|}$$

for $|\tau| \leq N-1$, but not for $|\tau| \geq N$. Hence we could replace s_τ by $\hat{s}_\tau^{(p)}$ for $|\tau| \leq N-1$ and assume $s_\tau = 0$ for $|\tau| \geq N$.

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Hence,

$$\begin{aligned}
\hat{S}^{(p)}(f) &= \sum_{\tau=-(N-1)}^{(N-1)} \hat{s}_\tau^{(p)} e^{-i2\pi f\tau} \\
&= \frac{1}{N} \sum_{\tau=-(N-1)}^{(N-1)} \sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|} e^{-i2\pi f\tau} \\
&= \frac{1}{N} \sum_{j=1}^{N}\sum_{k=1}^{N} X_j X_k e^{-i2\pi f(k-j)} = \frac{1}{N}\left|\sum_{t=1}^{N} X_t e^{-i2\pi ft}\right|^2,
\end{aligned}$$

$\hat{S}^{(p)}(f)$ defined above is known as the *periodogram*, and is defined over $[-1/2, 1/2]$.

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Note that $\{s_\tau^{(p)}\}$ and $\hat{S}^{(p)}(f)$,

$$\{s_\tau^{(p)}\} \longleftrightarrow \hat{S}^{(p)}(f)$$

just like the process quantities

$$\{s_\tau\} \longleftrightarrow S(f).$$

Hence, $\{s_\tau^{(p)}\}$ can be written as

$$s_\tau^{(p)} = \int_{-1/2}^{1/2} \hat{S}^{(p)}(f) e^{i2\pi f\tau}\, df \quad |\tau| \leq N-1.$$

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If $\hat{S}^{(p)}(f)$ were an ideal estimator of $S(f)$ we would have

i $E\{\hat{S}^{(p)}(f)\} \approx S(f) \quad \forall f.$

ii $Var\{\hat{S}^{(p)}(f)\} \to 0$ as $N \to \infty$ and,

iii $Cov\{\hat{S}^{(p)}(f), \hat{S}^{(p)}(f')\} \approx 0$ for $f \neq f'.$

We find that

i is a good approximation for some processes,

ii is patently false,

iii holds if f and f' are certain distinct frequencies, namely, the Fourier frequencies $f_k = k/N \quad (\Delta t = 1)$.

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We firstly look at the expectation in i. (assuming $\mu = 0$).

$$\begin{aligned} E\{\hat{S}^{(p)}(f)\} &= \sum_{\tau=-(N-1)}^{(N-1)} E\{s_\tau^{(p)}\} e^{-i2\pi f\tau} \\ &= \sum_{\tau=-(N-1)}^{(N-1)} \left(1 - \frac{|\tau|}{N}\right) s_\tau e^{-i2\pi f\tau}. \end{aligned}$$

Hence, if we know the acvs $\{s_\tau\}$ we can work out from this what $E\{\hat{S}^{(p)}(f)\}$ will be.

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We can obtain much more insight by considering:

$$E\{|J(f)|^2\} \quad \text{where} \quad J(f) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} X_t e^{-i2\pi ft}, \quad |f| \leq \frac{1}{2}.$$

as $\hat{S}^{(p)}(f) = |J(f)|^2$.

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We know from the spectral representation theorem that,

$$X_t = \int_{-1/2}^{1/2} e^{i2\pi f' t}\, dZ(f'),$$

so that,

$$\begin{aligned} J(f) &= \sum_{t=1}^{N} \left(\int_{-1/2}^{1/2} \frac{1}{\sqrt{N}} e^{i2\pi f' t}\, dZ(f') \right) e^{-i2\pi f t} \\ &= \int_{-1/2}^{1/2} \sum_{t=1}^{N} \frac{1}{\sqrt{N}} e^{-i2\pi (f-f')t}\, dZ(f') \end{aligned}$$

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We find that,

$$\begin{aligned} E\{\hat{S}^{(p)}(f)\} &= E\{|J(f)|^2\} = E\{J^*(f)J(f)\} \\ &= \int_{-1/2}^{1/2} \mathcal{F}(f-f')S(f')\,df', \end{aligned}$$

where $\mathcal{F}$ is Féjer's kernel defined by

$$\mathcal{F}(f) = \left| \sum_{t=1}^{N} \frac{1}{\sqrt{N}} e^{-i2\pi ft} \right|^2 = \frac{\sin^2(N\pi f)}{N\sin^2(\pi f)}.$$

This result tells us that the expected value of $\hat{S}^{(p)}(f)$ is the true spectrum convolved with Féjer's kernel.

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Properties of Féjer's kernel:

(a) For all integers $N \geq 1, \mathcal{F}(f) \to N$ as $f \to 0$.

(b) For $N \geq 1$, $f \in [-1/2, 1/2]$ and $f \neq 0$, $\mathcal{F}(f) < \mathcal{F}(0)$.

(c) For $f \in [-1/2, 1/2]$, $f \neq 0$, $\mathcal{F}(f) \to 0$ as $N \to \infty$.

(d) For any integer $k \neq 0$ such that $f_k = k/N \in [-1/2, 1/2]$, $\mathcal{F}(f_k) = 0$.

(e) $\int_{-1/2}^{1/2} \mathcal{F}(f)\, df = 1$.

$\mathcal{F}(f)$ is symmetric about the origin and consists of a broad central peak ("lobe") and $N - 2$ sidelobes which decrease as f increases. From (a), (c) and (e) it follows that as $N \to \infty$, $\mathcal{F}(f)$ acts as a *Dirac δ function*, with an infinite spike at $f = 0$.

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For a process with large dynamic range, defined as

$$10 \log_{10} \left(\frac{\max_f S(f)}{\min_f S(f)} \right)$$

as the expected value of the periodogram is a convolution of Féjer's kernel and the true spectrum, power from parts of the spectrum where $S(f)$ is large can "leak" via the sidelobes to other frequencies where $S(f)$ is small.

Session 3: Time Series Analysis

Bias reduction – Tapering

To reduce the bias in the periodogram we can use a technique called tapering.

Let $X_1, X_2, \ldots, X_N$ be a portion of length N of a zero mean stationary process with sdf $S(f)$. We form the product $\{h_t X_t\}$ where $\{h_t\}$ is a sequence of real-valued constants called a data taper . Define

$$J(f) = \sum_{t=1}^{N} h_t X_t e^{-i2\pi ft} \qquad |f| \leq 1/2.$$

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By the spectral representation theorem,

$$X_t = \int_{-1/2}^{1/2} e^{i2\pi f' t}\, dZ(f'),$$

so that,

$$\begin{aligned} J(f) &= \sum_{t=1}^{N} h_t \left(\int_{-1/2}^{1/2} e^{i2\pi f' t}\, dZ(f') \right) e^{-i2\pi f t} \\ &= \int_{-1/2}^{1/2} \sum_{t=1}^{N} h_t e^{-i2\pi (f-f')t}\, dZ(f') = \int_{-1/2}^{1/2} H(f-f')\, dZ(f'), \end{aligned}$$

where,

$$H(f) = \sum_{t=1}^{N} h_t e^{-i2\pi f t} \quad \text{i.e.,} \quad \{h_t\} \longleftrightarrow H(f).$$

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Let,

$$\hat{S}^{(d)}(f) = |J(f)|^2 = \left| \sum_{t=1}^{N} h_t X_t e^{-i2\pi ft} \right|^2 .$$

Then,

$$\begin{aligned} |J(f)|^2 &= J^*(f)J(f) \\ &= \int_{-1/2}^{1/2} H^*(f-f')\, dZ^*(f') \int_{-1/2}^{1/2} H(f-f'')\, dZ(f''). \end{aligned}$$

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Hence

$$\begin{aligned} E\{\hat{S}^{(d)}(f)\} &= E\{|J(f)|^2\} = \int_{-1/2}^{1/2} |H(f-f')|^2 S(f')\, df' \\ &= \int_{-1/2}^{1/2} \mathcal{H}(f-f') S(f')\, df', \end{aligned}$$

where $\mathcal{H}(f-f') = |H(f-f')|^2$, i.e.,

$$\mathcal{H}(f) = \left| \sum_{t=1}^{N} h_t e^{-i2\pi ft} \right|^2 .$$

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We take,

$$\sum_{t=1}^{N} h_t^2 = 1.$$

A spectral estimator of the form of $\hat{S}^{(d)}(f)$ is called a *direct spectral estimator* (hence the (d)).

Note, if $h_t = \frac{1}{\sqrt{N}}$ for $1 \leq t \leq N$, then

$$\hat{S}^{(d)}(f) = \hat{S}^{(p)}(f) \quad \text{and} \quad \mathcal{H}(f) = \mathcal{F}(f),$$

i.e., $\hat{S}^{(d)}(f)$ is the same as the periodogram, and $\mathcal{H}(f)$ is the same as Féjer's kernel.

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The key idea behind tapering is to select $\{h_t\}$ so that $\mathcal{H}(f)$ has much lower sidelobes that $\mathcal{F}(f)$. Recall that $\mathcal{F}(f)$ corresponds to a rectangular taper

$$h_t = \begin{cases} \frac{1}{\sqrt{N}} & \text{for } 1 \leq t \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

There is thus a sharp discontinuity between where the taper is "ON" $(1 \leq t \leq N)$ and where it is "OFF". Tapering effectively creates a smooth transition at the ends of the data.

Session 3: Time Series Analysis

Parametric model fitting

We focus on $AR(p)$ models, for which the sdf is

$$S(f) = \frac{\sigma^2}{|1 - \phi_{1,p} e^{-i2\pi f} - \ldots - \phi_{p,p} e^{-i2\pi fp}|^2}.$$

This class of models is appealing for several reasons.

(i) Any time series with a purely continuous sdf can be approximated well by an AR(p) model if p is large enough.

(ii) There exist efficient algorithms for fitting AR(p) models to time series.

(iii) Quite a few physical phenomena are reverberant and hence an AR model is naturally appropriate.

Session 3: Time Series Analysis

The Yule-Walker Method

We start by multiplying the defining equation by X_{t-k}:

$$X_t X_{t-k} = \sum_{j=1}^{p} \phi_{j,p} X_{t-j} X_{t-k} + \epsilon_t X_{t-k}.$$

Taking expectations, for $k > 0$:

$$s_k = \sum_{j=1}^{p} \phi_{j,p} s_{k-j}.$$

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Let $k = 1, 2, \ldots, p$ and recall that $s_{-\tau} = s_\tau$ to obtain

$$\begin{aligned} s_1 &= \phi_{1,p} s_0 + \phi_{2,p} s_1 + \ldots + \phi_{p,p} s_{p-1} \\ s_2 &= \phi_{1,p} s_1 + \phi_{2,p} s_0 + \ldots + \phi_{p,p} s_{p-2} \\ &\vdots \quad\quad \vdots \\ s_p &= \phi_{1,p} s_{p-1} + \phi_{2,p} s_{p-2} + \ldots + \phi_{p,p} s_0 \end{aligned}$$

or in matrix notation, $\gamma_p = \Gamma_p \phi_p$, where $\gamma_p = [s_1, s_2, \ldots, s_p]^T$, $\phi_p = [\phi_{1,p}, \phi_{2,p}, \ldots, \phi_{p,p}]^T$ and

$$\Gamma_p = \begin{bmatrix} s_0 & s_1 & \ldots & s_{p-1} \\ s_1 & s_0 & \ldots & s_{p-2} \\ \vdots & \vdots & & \vdots \\ s_{p-1} & s_{p-2} & \ldots & s_0 \end{bmatrix}$$

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Suppose we don't know the $\{s_\tau\}$, but the mean is zero, then take

$$\hat{s}_\tau = \frac{1}{N} \sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|},$$

and substitute these for the s_τ's in γ and Γ_p to obtain $\hat{\gamma}_p, \hat{\Gamma}_p$, from which we estimate ϕ_p as $\hat{\phi}_p$:

$$\hat{\phi}_p = \Gamma^{-1} \hat{\gamma}_p.$$

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Finally, we need to estimate σ_ϵ^2. To do so, we multiply the defining equation by X_t and take expectations to obtain

$$s_0 = \sum_{j=1}^{p} \phi_{j,p} s_j + E\{\epsilon_t X_t\} = \sum_{j=1}^{p} \phi_{j,p} s_j + \sigma_\epsilon^2,$$

so that as an estimator for σ_ϵ^2 we take

$$\hat{\sigma}_\epsilon^2 = \hat{s}_o - \sum_{j=1}^{p} \hat{\phi}_{j,p} \hat{s}_j.$$

The estimators $\hat{\phi}_p$ and $\hat{\sigma}_\epsilon^2$ are called the Yule-Walker estimators of the AR(p) process.

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The estimate of the sdf resulting is

$$\hat{S}(f) = \frac{\hat{\sigma}_\epsilon^2}{\left|1 - \sum_{j=1}^{p} \hat{\phi}_{j,p} e^{-i2\pi fj}\right|^2}.$$

There are important modifications which we can make to this approach: we could use for $\{\hat{s}_\tau\}$ a modified autocovariance incorporating tapering:

$$\hat{s}_\tau = \sum_{t=1}^{N-|\tau|} h_t X_t h_{t+|\tau|} X_{t+|\tau|}.$$

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Levinson-Durbin

To invert $\hat{\Gamma}_p$ by brute force matrix inversion requires $O(p^3)$ operations.

Fortunately, there is an algorithm due to Levinson and Durbin which takes advantage of the highly structured nature of the Toeplitz matrix, and carries out the estimation in $O(p^2)$ or fewer operations.

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Least squares estimation of the $\{\phi_{j,p}\}$

Let $\{X_t\}$ be a zero-mean AR(p) process, i.e.,

$$X_t = \phi_{1,p}X_{t-1} + \phi_{2,p}X_{t-2} + \ldots + \phi_{p,p}X_{t-p} + \epsilon_t.$$

We can formulate an appropriate least squares model in terms of data $X_1, X_2, \ldots, X_N$ as follows:

$$\mathbf{X}_F = F\boldsymbol{\phi} + \boldsymbol{\epsilon}_F,$$

where,

$$F = \begin{bmatrix} X_p & X_{p-1} & \ldots & X_1 \\ X_{p+1} & X_p & \ldots & X_2 \\ \vdots & & & \vdots \\ X_{N-1} & X_{N-2} & \ldots & X_{N-p} \end{bmatrix}$$

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and,

$$\mathbf{X}_F = \begin{bmatrix} X_{p+1} \\ X_{p+2} \\ \vdots \\ X_N \end{bmatrix}; \quad \boldsymbol{\phi} = \begin{bmatrix} \phi_{1,p} \\ \phi_{2,p} \\ \vdots \\ \phi_{p,p} \end{bmatrix}; \quad \boldsymbol{\epsilon}_F = \begin{bmatrix} \epsilon_{p+1} \\ \epsilon_{p+2} \\ \vdots \\ \epsilon_N \end{bmatrix}.$$

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We can thus estimate ϕ by finding that ϕ such that

$$\begin{aligned} \mathrm{SS}_F(\phi) &= \sum_{t=p+1}^{N} \left(X_t - \sum_{k=1}^{p} \phi_{k,p} X_{t-k} \right)^2 \qquad \left[= \sum_{t=p+1}^{N} \epsilon_t^2 \right] \\ &= (\mathbf{X}_F - F\phi)^T (\mathbf{X}_F - F\phi) \end{aligned}$$

is minimized. If we denote the vector that minimizes the above as $\hat{\phi}_F$, standard least squares theory tells us that it is given by

$$\hat{\phi}_F = (F^T F)^{-1} F^T \mathbf{X}_F.$$

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We can estimate the innovations variance σ_F^2 by the usual estimator of the residual variation, namely

$$\hat{\sigma}_F^2 = \frac{(\mathbf{X}_F - F\hat{\phi}_F)^T(\mathbf{X}_F - F\hat{\phi}_F)}{(N-2p)}.$$

(Note: there are $N-p$ effective observations, and p parameters are estimated).

The estimator $\hat{\phi}_F$ is known as the *forward* least squares estimator of ϕ.

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Using a *time reversed* formulation;

$$\mathbf{X}_B = B\phi + \boldsymbol{\epsilon}_B,$$

where,

$$B = \begin{bmatrix} X_2 & X_3 & \dots & X_{p+1} \\ X_3 & X_4 & \dots & X_{p+2} \\ \vdots & & & \vdots \\ X_{N-p+1} & X_{N-p+2} & \dots & X_N \end{bmatrix}$$

and,

$$\mathbf{X}_B = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-p} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\epsilon}_B = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_{N-p} \end{bmatrix}.$$

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The function of ϕ to be minimized is now

$$\begin{aligned} SS_B(\phi) &= \sum_{t=1}^{N-p} \left(X_t - \sum_{k=1}^{p} \phi_{k,p} X_{t+k} \right)^2 \\ &= (\mathbf{X}_B - B\phi)^T (\mathbf{X}_B - B\phi) \end{aligned}$$

The *backward* least squares estimator of ϕ is then given by

$$\hat{\phi}_B = (B^T B)^{-1} B^T \mathbf{X}_B.$$

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The corresponding estimator of the innovations variance σ_B^2 is

$$\hat{\sigma}_B^2 = \frac{(\mathbf{X}_B - B\phi)^T(\mathbf{X}_B - B\phi)}{(N-2p)}.$$

The vector $\hat{\phi}_{FB}$ that minimizes

$$SS_F(\phi) + SS_B(\phi)$$

is called the *forward/backward* least squares estimator, and Monte-Carlo studies indicate that it performs better than forward or backward least squares.

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Notes:

- $\hat{\phi}_{FB}, \hat{\phi}_B$ and $\hat{\phi}_F$ produce estimated models which need not be stationary. This may be a concern for prediction, however, for spectral estimation, the parameter values will still produce a valid sdf (i.e., nonnegative everywhere, symmetric about the origin and integrates to a finite number).
- The Yule-Walker estimates can be formulated as a least squares problem; consider adding zeros to our observations $X_1, X_2, \ldots, X_N$, both at the beginning and end of the data, to give:

$$\mathbf{X}_{YW} = W\phi + \epsilon_{YW},$$

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$$W = \begin{bmatrix} 0 & 0 & 0 & \dots & \dots & 0 \\ X_1 & 0 & 0 & \dots & \dots & 0 \\ X_2 & X_1 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & & & & \vdots \\ X_{p-1} & \vdots & & & & 0 \\ X_p & X_{p-1} & \dots & \dots & \dots & X_1 \\ \vdots & \vdots & & & & \vdots \\ X_N & X_{N-1} & \dots & \dots & \dots & X_{N-p+1} \\ 0 & X_N & & & & X_{N-p+2} \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & & & X_N \end{bmatrix}$$

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Therefore

$$\mathbf{X}_{YW} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\epsilon}_{YW} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_N \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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$$\frac{1}{N} W^T W = \begin{bmatrix} \hat{s}_0^{(p)} & \hat{s}_1^{(p)} & \cdots & \hat{s}_{p-1}^{(p)} \\ \hat{s}_1^{(p)} & \ddots & & \\ \vdots & \ddots & \ddots & \\ \hat{s}_{p-1}^{(p)} & \cdots & \cdots & \hat{s}_0^{(p)} \end{bmatrix} = \hat{\Gamma}_p$$

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and

$$\frac{1}{N} W^T \mathbf{X}_{YW} = \begin{bmatrix} \hat{s}_1^{(p)} \\ \vdots \\ \hat{s}_p^{(p)} \end{bmatrix} = \hat{\gamma}_p,$$

so that

$$(W^T W)^{-1} W^T \mathbf{X}_{YW} = (\hat{\Gamma}_p)^{-1} \hat{\gamma}_p.$$

which is identical to the Yule-Walker estimate.

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Forecasting

Suppose we wish to predict the value of X_{t+l} of a process, given $X_t, X_{t-1}, X_{t-2}, \ldots$. Let the appropriate model for $\{X_t\}$ be an ARMA(p, q) process:

$$\Phi(B)X_t = \Theta(B)\epsilon_t.$$

Consider a forecast $X_t(l)$ of X_{t+1} (an l-step ahead forecast) which is a linear combination of $X_t, X_{t-1}, X_{t-2}, \ldots$:

$$X_t(l) = \sum_{k=0}^{\infty} \pi_k X_{t-k}.$$

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Note: this assumes a semi-infinite realization of $\{X_t\}$. Let us now assume that $\{X_t\}$ can be written as a one-sided linear process, so that

$$X_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k} = \Psi(B)\epsilon_t,$$

and

$$X_{t+l} = \sum_{k=0}^{\infty} \psi_k \epsilon_{t+l-k} = \Psi(B)\epsilon_{t+l}.$$

Hence,

$$X_t(l) = \sum_{k=0}^{\infty} \pi_k X_{t-k} \quad = \quad \sum_{k=0}^{\infty} \pi_k \Psi(B)\epsilon_{t-k} = \Pi(B)\Psi(B)\epsilon_t.$$

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Let $\delta(B) = \Pi(B)\Psi(B)$ so that,

$$X_t(l) = \delta(B)\epsilon_t = \sum_{k=0}^{\infty} \delta_k \epsilon_{t-k}.$$

Now,

$$X_{t+l} = \sum_{k=0}^{\infty} \psi_k \epsilon_{t+l-k} \quad = \quad \sum_{k=0}^{l-1} \psi_k \epsilon_{t+l-k} + \sum_{k=l}^{\infty} \psi_k \epsilon_{t+l-k} = (\mathrm{A}) + (\mathrm{B})$$

(A) Involves future ϵ_ts, represents the "unpredictable" part of X_{t+l}.

(B) Depends only on past and present values of ϵ_t, represents the "predictable" part of X_{t+l}.

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Hence we would expect

$$X_t(l) = \sum_{k=l}^{\infty} \psi_k \epsilon_{t+l-k} = \sum_{j=0}^{\infty} \psi_{j+l} \epsilon_{t-j}.$$

so that $\delta_k \equiv \psi_{k+l}$. This can be readily proved. For linear least squares, we want to minimize,

$$\begin{aligned} E\{(X_{t+l} - X_t(l))^2\} &= E\left\{ \left(\sum_{k=0}^{l-1} \psi_k \epsilon_{t+l-k} + \sum_{k=0}^{\infty} [\psi_{k+l} - \delta_k] \epsilon_{t-k} \right)^2 \right\} \\ &= \sigma_\epsilon^2 \left\{ \left(\sum_{k=0}^{l-1} \psi_k^2 \right) + \sum_{k=0}^{\infty} (\psi_{k+l} - \delta_k)^2 \right\}. \end{aligned}$$

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The first term is independent of the choice of the $\{\delta_k\}$ and the second term is clearly minimized by choosing $\delta_k = \psi_{k+l}, k = 0, 1, 2, \ldots$ as expected. With this choice of $\{\delta_k\}$ the second term vanishes, and we have,

$$\begin{aligned} \sigma^2(l) &= E\{(X_{t+l} - X_t(l))^2\} \\ &= \sigma_\epsilon^2 \sum_{k=0}^{l-1} \psi_k^2, \end{aligned}$$

which is known as the l-step prediction variance.

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When $l = 1$, $\delta_k = \psi_{k+1}$,

$$\begin{aligned} X_t(1) &= \delta_0\epsilon_t + \delta_1\epsilon_{t-1} + \delta_2\epsilon_{t-2} + \ldots \\ &= \psi_1\epsilon_t + \psi_2\epsilon_{t-1} + \psi_3\epsilon_{t-2} + \ldots \\ X_{t+1} &= \psi_0\epsilon_{t+1} + \psi_1\epsilon_t + \psi_2\epsilon_{t-1} + \ldots \end{aligned}$$

so that,

$$X_{t+1} - X_t(1) = \psi_0\epsilon_{t+1} = \epsilon_{t+1} \quad \text{since} \quad \psi_0 = 1.$$

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Hence ϵ_{t+1} can be thought of as the "one step prediction error". Also of course,

$$X_{t+1} = X_t(1) + \epsilon_{t+1}$$

so that ϵ_{t+1} is the essentially "new" part of X_{t+1} which is not linearly dependent on past observations. The sequence $\{\epsilon_t\}$ is often called the innovations process of $\{X_t\}$, and σ^2_ϵ is often called the innovations variance.

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If we wish to write $X_t(l)$ explicitly as a function of $X_t, X_{t-1}, \ldots$ then we note first that,

$$X_t(l) = \sum_{k=0}^{\infty} \delta_k \epsilon_{t-k} = \sum_{k=0}^{\infty} \psi_{k+l} \epsilon_{t-k},$$

so that,

$$X_t(l) = \Psi^{(l)}(B)\epsilon_t, \qquad \text{say}$$

where,

$$\Psi^{(l)}(z) = \sum_{k=0}^{\infty} \psi_{k+l} z^k.$$

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Assuming that $\Psi(z)$ is analytic in and on the unit circle (stationary and invertible) then we can write

$$X_t = \Psi(B)\epsilon_t \quad \text{and} \quad \epsilon_t = \Psi^{-1}(B)X_t,$$

and thus

$$\begin{aligned} X_t(l) = \Psi^{(l)}(B)\epsilon_t &= \Psi^{(l)}(B)\Psi^{-1}(B)X_t \\ &= G^{(l)}(B)X_t, \quad \text{say} \end{aligned}$$

with,

$$G^{(l)}(z) = \Psi^{(l)}(z)\Psi^{-1}(z).$$

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If we consider the sequence of predictors $X_t(l)$ for different values of t (with l fixed) then this forms a new process, which since

$$X_t(l) = G^{(l)}(B)X_t,$$

may be regarded as the output of a linear filter acting on the $\{X_t\}$. Since,

$$X_t(l) = \left(\sum_u g_u^{(l)} B^u\right) X_t = \sum_u g_u^{(l)} X_{t-u},$$

we know that the transfer function is

$$G^{(l)}(f) = \sum_u g_u^{(l)} e^{-i2\pi fu}.$$

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Example: AR(1)

$$X_t - \phi_{1,1} X_{t-1} = \epsilon_t \qquad |\phi_{1,1}| < 1.$$

Then

$$X_t = (1 - \phi_{1,1} B)^{-1} \epsilon_t.$$

So,

$$\begin{aligned} \Psi(z) &= 1 + \phi_{1,1} z + \phi_{1,1}^2 z^2 + \ldots \\ &= \psi_0 + \psi_1 z + \psi_2 z^2 + \ldots \end{aligned}$$

i.e., $\psi_k = \phi_{1,1}^k$.

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Hence,

$$\begin{aligned} X_t(l) &= \sum_{k=0}^{\infty} \delta_k \epsilon_{t-k} = \sum_{k=0}^{\infty} \psi_{k+l} \epsilon_{t-k} \\ &= \sum_{k=0}^{\infty} \phi_{1,1}^{k+l} \epsilon_{t-k} = \phi_{1,1}^{l} \sum_{k=0}^{\infty} \phi_{1,1}^{k} \epsilon_{t-k} \\ &= \phi_{1,1}^{l} X_t. \end{aligned}$$

The l-step prediction variance is

$$\sigma^2(l) = \sigma_\epsilon^2 \left(\sum_{k=0}^{l-1} \psi_k^2 \right) = \sigma_\epsilon^2 \left(\sum_{k=0}^{l-1} \phi_{1,1}^{2k} \right) = \sigma_\epsilon^2 \frac{(1-\phi_{1,1}^{2l})}{(1-\phi_{1,1}^2)}.$$

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Alternatively,

$$X_t(l) = G^{(l)}(B)X_t,$$

with $G^{(l)}(z) = \Psi^{(l)}(z)\Psi^{-1}(z)$. But,

$$\Psi^{(l)}(z) = \sum_{k=0}^{\infty} \psi_{k+l} z^k = \sum_{k=0}^{\infty} \phi_{1,1}^{k+l} z^k,$$

and,

$$\Psi^{-1}(z) = 1 - \phi_{1,1} z,$$

so that

$$G^{(l)}(z) = (\phi_{1,1}^{l} + \phi_{1,1}^{l+1} z + \phi_{1,1}^{l+2} z^2 + \ldots)(1 - \phi_{1,1} z) = \phi_{1,1}^{l},$$

i.e., $X_t(l) = \phi_{1,1}^{l} X_t$ as before.

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We have demonstrated that for the AR(1) model the linear least squares predictor of X_{t+l} depends only on the most recent observation, X_t, and does not involve $X_{t-1}, X_{t-2}, \ldots$, which is what we would expect bearing in mind the Markov nature of the AR(1) model. As $l \to \infty$, $X_t(l) \to 0$, since $X_t(l) = \phi_{1,1}^l X_t$ and $|\phi_{1,1}| < 1$. Also, the l-step prediction variance,

$$\sigma^2(l) \to \frac{\sigma_\epsilon^2}{(1-\phi_{1,1}^2)} = Var\left[X_t\right].$$

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In fact the solution to the forecasting problem for the AR(1) model can be derived directly from the difference equation,

$$X_t - \phi_{1,1} X_{t-1} = \epsilon_t.$$

by setting future innovations ϵ_t to be zero:

$$\begin{aligned} X_t(1) &= \phi_{1,1} X_t + 0 \\ X_t(2) &= \phi_{1,1} X_t(1) + 0 \\ &\vdots \\ X_t(l) &= \phi_{1,1} X_t(l-1) + 0 \end{aligned}$$

so that,

$$X_t(l) = \phi_{1,1}^l X_t.$$

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For general AR(p) processes it turns out that $X_t(l)$ depends only on the last p observed values of $\{X_t\}$, and may be obtained by solving the AR(p) difference equation with the future $\{\epsilon_t\}$ set to zero. For example for an AR(p) process and $l = 1$,

$$X_t(1) = \phi_{1,p} X_t + \ldots + \phi_{p,p} X_{t-p+1}.$$

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Example: ARMA(1,1)

$$(1 - \phi_{1,1}B)X_t = (1 - \theta_{1,1}B)\epsilon_t.$$

Take $\phi_{1,1} = \phi$ and $\theta_{1,1} = \theta$,

$$X_t = \frac{(1 - \theta B)}{(1 - \phi B)}\epsilon_t = \Psi(B)\epsilon_t.$$

So,

$$\begin{aligned}
\Psi(z) &= (1 - \theta z)(1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \ldots) \\
&= 1 + (\phi - \theta)z + \phi(\phi - \theta)z^2 + \ldots + \phi^{l-1}(\phi - \theta)z^l + \ldots \\
&= \psi_0 + \psi_1 z + \psi_2 z^2 + \ldots
\end{aligned}$$

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So,

$$\psi_l = \begin{cases} 1 & l = 0 \\ \phi^{l-1}(\phi - \theta) & l \geq 1 \end{cases}$$

The l-step prediction variance is

$$\begin{aligned} \sigma^2(l) &= \sigma_\epsilon^2 \left(\sum_{k=0}^{l-1} \psi_k^2 \right) = \sigma_\epsilon^2 \left(1 + \sum_{k=1}^{l-1} \psi_k^2 \right) \\ &= \sigma_\epsilon^2 \left(1 + (\phi - \theta)^2 \sum_{k=1}^{l-1} \phi^{2k-2} \right) \\ &= \sigma_\epsilon^2 \left(1 + (\phi - \theta)^2 \frac{(1 - \phi^{2l-2})}{(1 - \phi^2)} \right). \end{aligned}$$

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Now,

$$\begin{aligned}\psi^{(l)}(z) &= \sum_{k=0}^{\infty}\psi_{k+l}z^k = \phi^{l-1}(\phi-\theta)\sum_{k=0}^{\infty}\phi^k z^k = \phi^{l-1}(\phi-\theta)(1-\phi z)^{-1}\\ \psi^{-1}(z) &= \frac{(1-\phi z)}{(1-\theta z)},\end{aligned}$$

so therefore

$$\begin{aligned}G^{(l)}(z) &= \psi^{(l)}(z)\psi^{-1}(z) = \phi^{l-1}(\phi-\theta)(1-\theta z)^{-1}\\ X_t(l) &= G^{(l)}(B)X_t = \phi^{l-1}(\phi-\theta)(1-\theta B)^{-1}X_t.\end{aligned}$$

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Consider $l = 1$,

$$\begin{aligned} X_t(1) &= (\phi - \theta)(1 - \theta B)^{-1} X_t \\ &= (\phi - \theta)(1 + \theta B + \theta^2 B^2 + \theta^3 B^3 + \ldots) X_t \\ &\vdots \\ &= (\phi - \theta) X_t + \theta(\phi - \theta) X_{t-1} + \theta^2 (\phi - \theta) X_{t-2} + \ldots \\ &= \phi X_t - \theta \left[X_t - (\phi - \theta) X_{t-1} - \ldots - \theta^{k-1} (\phi - \theta) X_{t-k} - \ldots \right] \end{aligned}$$

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But consider,

$$\begin{aligned}
\epsilon_t &= \Psi^{-1}(B)X_t = (1-\phi B)(1-\theta B)^{-1}X_t \\
&= (1-\phi B)(1+\theta B+\theta^2 B^2+\theta^3 B^3+\ldots)X_t \\
&\vdots \\
&= X_t-(\phi-\theta)X_{t-1}-\ldots-\theta^{k-1}(\phi-\theta)X_{t-k}-\ldots.
\end{aligned}$$

Therefore,

$$X_t(1) = \phi X_t - \theta\epsilon_t.$$

So can again be derived directly from the difference equation,

$$X_t = \phi X_{t-1} - \theta\epsilon_{t-1} + \epsilon_t,$$

by setting future innovations ϵ_t to zero.

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MA(1) (invertible)

$$X_t = \epsilon_t - \theta_{1,1}\epsilon_{t-1} \qquad |\theta_{1,1}| < 1.$$

So,

$$\begin{aligned} \Psi(z) &= \psi_0 + \psi_1 z + \psi_2 z^2 + \ldots \\ &= 1 - \theta_{1,1} z \end{aligned}$$

Hence, $\psi_0 = 1; \quad \psi_1 = -\theta_{1,1}; \quad \psi_k = 0, \quad k \geq 2.$

$$\begin{aligned} X_t(l) &= \sum_{k=0}^{\infty} \psi_{k+l}\epsilon_{t-k} = \Psi^{(l)}(B)\epsilon_t \\ &= \psi_l \epsilon_t + \psi_{l+1}\epsilon_{t-1} + \ldots \end{aligned}$$

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So,

$$\begin{aligned}\psi^{(l)}(z) &= \sum_{k=0}^{\infty} \psi_{k+l} z^k = \psi_l z^0 + \psi_{l+1} z^1 \\ &= \begin{cases} -\theta_{1,1} & l = 1 \\ 0 & l \geq 2. \end{cases}\end{aligned}$$

Hence,

$$G^{(l)}(z) = \psi^{(l)}(z)\psi^{-1}(z) = \begin{cases} -\theta_{1,1}(1-\theta_{1,1}z)^{-1} & l = 1 \\ 0 & l \geq 2. \end{cases}$$

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Thus, for $l = 1$,

$$G^{(1)}(z) = -\theta_{1,1}(1 + \theta_{1,1}z + \theta_{1,1}^2 z^2 + \ldots),$$

and hence,

$$X_t(1) \quad = \quad G^{(1)}(B)X_t = -\sum_{k=0}^{\infty} \theta_{1,1}^{k+1} X_{t-k}$$

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Forecast errors and updating

We have seen that when $\delta_k = \psi_{k+l}$ the forecast error is

$$\sum_{k=0}^{l-1} \psi_k \epsilon_{t+l-k}.$$

Let,

$$e_t(l) \quad = \quad X_{t+l} - X_t(l) = \sum_{k=0}^{l-1} \psi_k \epsilon_{t+l-k}.$$

Then,

$$e_t(l+m) = \sum_{j=0}^{l+m-1} \psi_j \epsilon_{t+l+m-j}.$$

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Clearly,

$$E\{e_t(l)\} = E\{e_t(l+m)\} = 0.$$

Hence,

$$Cov\{e_t(l), e_t(l+m)\} = E\{e_t(l)e_t(l+m)\} = \sigma_\epsilon^2 \sum_{k=0}^{l-1} \psi_k \psi_{k+m}.$$

and

$$Var\{e_t(l)\} = \sigma_\epsilon^2 \sum_{k=0}^{l-1} \psi_k^2 = \sigma^2(l).$$

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E.g.,

$$Cov\{e_t(1), e_t(2)\} = \sigma_\epsilon^2 \psi_1.$$

This could be quite large – should the forecast for a series wander of target, it is possible for it to remain there in the short run since forecast errors can be quite highly correlated. Hence, when X_{t+1} becomes available we should update the forecast.

$$\begin{aligned} X_{t+1}(l) &= \sum_{k=0}^{\infty} \psi_{k+l} \epsilon_{t+1-k} \\ &= \psi_l \epsilon_{t+1} + \psi_{l+1} \epsilon_t + \psi_{l+2} \epsilon_{t-1} + \ldots, \end{aligned}$$

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$$\begin{aligned} X_t(l+1) &= \sum_{k=0}^{\infty} \psi_{k+l+1}\epsilon_{t-k} \\ &= \psi_{l+1}\epsilon_t + \psi_{l+2}\epsilon_{t-1} + \psi_{l+3}\epsilon_{t-2} + \ldots, \end{aligned}$$

and,

$$\begin{aligned} X_{t+1}(l) &= X_t(l+1) + \psi_l\epsilon_{t+1} \\ &= X_t(l+1) + \psi_l(X_{t+1} - X_t(1)). \end{aligned}$$

Hence, to forecast X_{t+l+1} we can modify the $l+1$- step ahead forecast at time t by producing an l-step ahead forecast at time $t+1$ using X_{t+1} as it becomes available.

Session 3: Time Series Analysis

Non-stationarity and Unit Roots

Many financial/econometric series are *trending*.

Two cases commonly considered;

1. Stationary process with **deterministic** trend (shocks have temporary effects)
2. Process with **stochastic** trend or **unit root** (shocks have permanent effects)

The distinction between the two cases is practically important for forecasting and statistical issues.

Trend Stationarity

Example: Consider an AR(1) model with *deterministic linear trend*

$$Y_t = \phi Y_{t-1} + \delta + \gamma t + \epsilon_t \qquad t = 1, \ldots, N,$$

with $|\phi| < 1$. Then, as $N \longrightarrow \infty$,

$$E[Y_t] \longrightarrow \mu + \mu_1 t \qquad Var[Y_t] \longrightarrow \frac{\sigma^2}{1-\phi^2}$$

using the MA representation.

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- Y_t is not stationary, but the deviation from the mean

$$X_t = Y_t - \mu - \mu_1 t$$

  is stationary; Y_t is termed *trend-stationary*.
- The stochastic part is stationary, and shocks have transitory effects.
- Y_t is *mean-reverting*, with *attractor* $\mu + \mu_1 t$.

We can analyze X_t as a stationary process.

Unit Root Processes

Example: Consider an AR(1) model with a *unit root* $\phi = 1$

$$Y_t = Y_{t-1} + \delta + \epsilon_t$$

or

$$BY_t = \delta + \epsilon_t.$$

- $z = 1$ is a root of the AR polynomial $\Phi(z) = 1 - z$.
- Y_t is non-stationary.
- BY_t is stationary, Y_t termed a *difference stationary process*.
- Y_t is termed an *integrated first order process*, or an $I(1)$ process.
- A process of *integrated order* d is denoted $I(d)$.

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Note that

$$Y_t = Y_0 + \sum_{i=1}^{t} BY_t = Y_0 + \delta t + \sum_{i=1}^{t} \epsilon_t$$

with moments

$$E[Y_t] = Y_0 + \delta t \qquad V[Y_t] = t\sigma^2$$

- Y_0 remains in the process.
- ϵ_t accumulates as a random walk, termed a *stochastic trend*. These shocks have a permanent effect.
- δ forms a deterministic linear trend.
- This model is termed a *random walk with drift*.
- Variance grows with t.
- Not mean-reverting.

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Unit Root Tests

We consider null and alternative hypotheses to distinguish between stationarity and non-stationarity.

(1) Dickey-Fuller Test

- H_0 is a unit root, H_1 is stationarity

(2) KPSS Test

- H_0 is stationarity, H_1 is a unit root

Note: In practice, distinguishing $\phi = 0.99$ from $\phi = 1$ is often difficult ...

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Dickey-Fuller Test Set up an AR model for de-trended process X_t and test $\phi = 1$.

- Consider AR(1) model

$$X_t = \theta X_{t-1} + \epsilon_t$$

We wish to test

$$H_0 : \phi = 1 \qquad \text{against} \qquad H_1 : \phi < 1.$$

- Rewrite model as

$$BX_t = (\phi - 1)X_{t-1} + \epsilon_t = \pi X_{t-1} + \epsilon_t$$

with $\pi = \phi - 1 = \Phi(1)$, say, and the hypotheses as

$$H_0 : \pi = 0 \qquad \text{against} \qquad H_1 : \pi < 0.$$

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The Dickey-Fuller (DF) test is the Wald t-test for H_0 with test statistic t_{DF}

$$t_{DF} = \frac{\widehat{\phi} - 1}{se(\widehat{\phi})} = \frac{\widehat{\pi}}{se(\widehat{\pi})}$$

- The asymptotic null distribution is non-normal, and depends on the deterministic part of the model.
- The asymptotic null only holds if ϵ_t are IID.
- If not IID, need to include further terms in AR representation.
- MA and ARMA models handled similarly.

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Extension to AR(p): The **Augmented Dickey-Fuller** (ADF) Test.

Example: AR(3).

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + \epsilon_t$$

A unit root of

$$\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3 = 0$$

corresponds to $\Phi(1) = 0$.

Test is achieved by rewriting the model as

$$BX_t = \pi X_{t-1} + c_1 BX_{t-1} + c_2 BX_{t-2} + \epsilon_t$$

where

$$\begin{aligned} \pi &= \phi_1 + \phi_2 + \phi_3 - 1 = -\phi(1) \\ c_1 &= -(\phi_2 + \phi_3) \end{aligned}$$

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- Null hypothesis $\Phi(1) = 0$ corresponds to

$$H_0 : \pi = 0 \qquad \text{against} \qquad H_1 : \pi < 0.$$

- The ADF test is the Wald t-test of this hypothesis.
- Need model selection to choose number of lags.
- Can correct for autocorrelation in ϵ_t - use the *Phillips-Perron* test that uses a standard ergodic estimate of the autocorrelation (*Newey-West*).

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Note: The deterministic terms in the ADF specification are important, as they influence the asymptotic null distribution.

- if X_t has a non-zero level, use

$$BY_t = \pi Y_{t-1} + c_1 BX_{t-1} + c_2 BX_{t-2} + \delta\epsilon_t$$

- if X_t has a deterministic trend level, use

$$BY_t = \pi Y_{t-1} + c_1 BX_{t-1} + c_2 BX_{t-2} + \delta + \gamma t + \epsilon_t$$

In both cases, can fit model using regression methods.

In both cases, the null distribution changes.

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Note: consider the factor representation

$$\begin{aligned} X_t &= \phi X_{t-1} + \epsilon_t \\ Y_t &= X_t + \mu \end{aligned}$$

so that

$$Y_t = \phi Y_{t-1} + (1-\phi)\mu + \epsilon_t = \phi Y_{t-1} + \delta + \epsilon_t$$

so there is a common factor restriction; if $\phi = 1$,

$$\delta = (1-\phi)\mu = 0.$$

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This is not imposed by the standard t-test; consider

$$Y_t = \phi Y_{t-1} + \delta + \epsilon_t.$$

The hypotheses

$$H_0 : \phi = 1 \qquad \text{against} \qquad H_1 : \phi < 1.$$

imply

$$\begin{aligned} H_1 &: Y_t = \mu + \text{ stationary process} \\ H_0 &: Y_t = Y_0 + \delta t + \text{ stochastic trend.} \end{aligned}$$

that is, two fundamentally different models.

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Need to consider the combined null hypothesis

$$H_0^C : \pi = \delta = 0$$

which can be tested by fitting two regressions

$$\begin{aligned} H_1 &: BY_t = \pi Y_{t-1} + \delta + \epsilon_t \\ H_0^C &: BY_t = \epsilon_t. \end{aligned}$$

and carrying out a likelihood ratio test to compare the fits.

Again, the null distribution is non-standard.

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Alternatively, consider the model with a trend

$$BY_t = \pi Y_{t-1} + \delta + \gamma t + \epsilon_t$$

where the common factor restriction implies that if $\pi = 0$ then $\gamma = 0$. Under the standard null H_0, the trend will accumulate.

Again need to impose the combined null hypothesis

$$H_0^C : \pi = \gamma = 0$$

which can be tested by fitting two regressions

$$\begin{aligned} H_1 &: BY_t = \pi Y_{t-1} + \delta + \gamma t \epsilon_t \\ H_0^C &: BY_t = \delta + \epsilon_t. \end{aligned}$$

and carrying out a likelihood ratio test to compare the fits.

Again, the null distribution is non-standard.

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Special Events: Large shocks (breaks, changepoints) have potentially large, permanent effects.

- **One large shock**: may lead to bias toward accepting unit root hypothesis, event of series is stationary.
- **Many large shocks**: may lead to bias toward accepting stationarity hypothesis. Series may appear mean-reverting even if it is not.

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Kwiatkowski, Phillips, Schmidt and Shin (KPSS) Test

- Assume

$$Y_t = \xi_t + e_t$$

where e_t is stationary and ξ_t is a random walk

$$\xi_t = \xi_{t-1} + v_t$$

where $v_t \sim N(0, \sigma_v^2)$ i.i.d..

- If $\sigma_v^2 = 0$, $\xi_t = \xi_0$ and Y_t is stationary. Thus can test the hypothesis

$$H_0 : \sigma_v^2 = 0 \qquad \text{against} \qquad H_1 : \sigma_v^2 > 0.$$

The KPSS Test is a (score) test of this hypothesis.

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Models For Changing Variance

Objective: obtain better estimates of local variance.

p'th order ARCH(p)

ARCH stands for **autoregressive conditionally heteroscedastic**

Assume we have a derived time series $\{Y_t\}$ that is (approximately) uncorrelated but has a variance (volatility) that changes through time,

$$Y_t = \sigma_t \varepsilon_t \tag{1}$$

where $\{\varepsilon_t\}$ is a white noise sequence with zero mean and unit variance.

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Here, σ_t represents the local conditional standard deviation of the process. Note that σ_t is not observable directly.

$\{Y_t\}$ is ARCH(p) if it satisfies equation (1) and

$$\sigma_t^2 = \alpha + \beta_{1,p} y_{t-1}^2 + \ldots + \beta_{p,p} y_{t-p}^2, \tag{2}$$

where $\alpha > 0$ and $\beta_{j,p} \geq 0, j = 1, \ldots, p$ (to ensure the variance remains positive), and y_{t-1} is the observed value of the derived time series at time $(t-1)$

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Note

(a) the absence of the error term in equation (2).

(b) unconstrained estimation often leads to violation of the non-negativity constraints that are needed to ensure positive variance.

(c) quadratic form (i.e. modelling σ_t^2) prevents modelling of asymmetry in volatility (i.e. volatility tends to be higher after a decrease than after an equal increase and ARCH cannot account for this).

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ARCH(1)

$$\sigma_t^2 = \alpha + \beta_{1,1} y_{t-1}^2$$

Define, $v_t = y_t^2 - \sigma_t^2 \Rightarrow \sigma_t^2 = y_t^2 - v_t$. The model can also be written:

$$y_t^2 = \alpha + \beta_{1,1} y_{t-1}^2 + v_t,$$

i.e. an AR(1) model for $\{y_t^2\}$ where the errors, $\{v_t\}$, have zero mean, but as $v_t = \sigma_t^2(\epsilon_t^2 - 1)$ the errors are heteroscedastic.

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(p, q)'th order generalized autoregressive conditionally heteroscedastic model GARCH(p, q)

$\{Y_t\}$ is GARCH(p, q) if it satisfies equation (1) and

$$\sigma_t^2 = \alpha + \beta_{1,p} y_{t-1}^2 + \ldots + \beta_{p,p} y_{t-p}^2 + \gamma_{1,q} \sigma_{t-1}^2 + \ldots \gamma_{q,q} \sigma_{t-q}^2,$$

where the parameters are chosen to ensure positive variance.

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Stochastic volatility models SV

Stochastic volatility models treat σ_t as an unobserved random variable which is assumed to follow a certain stochastic process. The specification for the derived series $\{Y_t\}$ is:

$$Y_t = \sigma_t \varepsilon_t, \qquad \sigma_t^2 = \exp(h_t),$$

where ε_t is white noise with zero mean and unit variance, and let h_t, for example, be an AR(1) process:

$$h_t = \alpha + \beta_{1,1} h_{t-1} + \eta_t,$$

where $\{\eta_t\}$ is a white noise process with variance σ_η^2. If $|\beta_{1,1}| < 1$, h_t is stationary $\Rightarrow$ Y_t stationary.

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Notes:

(a) unlike the GARCH specification, h_t (which defines in turn σ_t) is NOT deterministic.

(b) the exponential specification ensures positive conditional variance.

(c) can be further generalized by assuming, for example, h_t follows an ARMA(p, q) model.

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Harmonic with additive white noise
Here $\{X_t\}$ is expressed as

$$X_t = \cos(2\pi f_0 t + \phi) + \epsilon_t$$

f_0 is a fixed frequency and $\{\epsilon_t\}$ is zero mean white noise with variance σ^2_ϵ.

Case (a) ϕ is constant.

$$E[X_t] = E[\cos(2\pi f_0 t + \phi)] + E[\epsilon_t] = \cos(2\pi f_0 t + \phi).$$

so, mean depends on $t \Rightarrow$ not stationary.

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Case (b): $\phi \sim U[-\pi, \pi]$ and independent of $\{\epsilon_t\}$.

$$E[X_t] = E[\cos(2\pi f_0 t + \phi) + \epsilon_t] = E\{\cos(2\pi f_0 t + \phi)\}$$

Now,

$$\begin{aligned} E\{\cos(2\pi f_0 t + \phi)\} &= \int_{-\pi}^{\pi} \cos(2\pi f_0 t + \phi) \frac{1}{2\pi}\, d\phi \\ &= \left[\frac{\sin(2\pi f_0 t + \phi)}{2\pi}\right]_{-\pi}^{\pi} = 0. \end{aligned}$$

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So $E[X_t] = 0$, and, using the fact that $\{e_t\}$ and ϕ are independent.

$$\begin{aligned} E[X_t X_{t+\tau}] &= E[[\cos(2\pi f_0 t + \phi) + \epsilon_t][\cos(2\pi f_0(t+\tau) + \phi) + \epsilon_{t+\tau}]] \\ &= E[\cos(2\pi f_0 t + \phi)\cos(2\pi f_0 t + \phi + 2\pi f_0 \tau)] + E[\epsilon_t \epsilon_{t+\tau}]. \end{aligned}$$

Recall, as $\{\epsilon_t\}$ is white noise we have,

$$E\{\epsilon_t \epsilon_{t+\tau}\} = \begin{cases} \sigma_\epsilon^2 & \text{if } \tau = 0, \\ 0 & \text{if } \tau \neq 0, \end{cases}$$

So, for $\tau = 0$,

$$Cov\{X_t, X_t\} = s_0 = E\{\cos^2(2\pi f_0 t + \phi)\} + \sigma_\epsilon^2.$$

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Now,

$$\begin{aligned} E\{\cos^2(2\pi f_0 t + \phi)\} &= \int_{-\pi}^{\pi} \cos^2(2\pi f_0 t + \phi) \frac{1}{2\pi}\, d\phi \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [1 + \cos(4\pi f_0 t + 2\phi)] \frac{1}{2\pi}\, d\phi = \frac{1}{2}. \end{aligned}$$

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So, $s_0 = \frac{1}{2} + \sigma_\epsilon^2$, and for $\tau > 0$,

$$\begin{aligned} Cov\left[X_t, X_{t+\tau}\right] &= s_\tau = E\left[\cos(2\pi f_0 t + \phi)\cos(2\pi f_0 t + \phi + 2\pi f_0 \tau)\right] \\ &= \frac{1}{2}E\left[\cos(4\pi f_0 t + 2\phi + 2\pi f_0 \tau) + \cos(2\pi f_0 \tau)\right] \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \cos(2\pi f_0 \tau)\frac{1}{2\pi}\, \mathrm{d}\phi \\ &= \frac{\cos(2\pi f_0 \tau)}{2}\left[\frac{\phi}{2\pi}\right]_{-\pi}^{\pi} = \frac{\cos(2\pi f_0 \tau)}{2} \end{aligned}$$

which does not depend on $t \Rightarrow X_t$ is stationary.

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Trend removal and seasonal adjustment

There are certain, quite common, situations where the observations exhibit a trend – a tendency to increase or decrease slowly steadily over time – or may fluctuate in a periodic manner due to seasonal effects. The model is modified to

$$X_t = \mu_t + Y_t$$

- μ_t = time dependent mean.
- Y_t = zero mean stationary process.

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Trend adjustment for CO^2 data: $\{X_t\}$ is monthly atmospheric CO^2 concentrations expressed in parts per million (ppm) derived from in situ air samples collected at Mauna Loa observatory, Hawaii. Monthly data from May 1988 – December 1998, giving $N = 128$. Model suggested by plot:

$$X_t = \alpha + \beta t + Y_t.$$

(a) Estimate α and β by least squares, and work with the residuals

$$\hat{Y}_t = X_t - \hat{\alpha} - \hat{\beta} t.$$

(b) Take first differences:

$$X_t^{(1)} = X_t - X_{t-1} = \alpha + \beta t + Y_t - (\alpha + \beta(t-1) + Y_{t-1}) = \beta + Y_t - Y_{t-1}.$$

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Note: if $\{Y_t\}$ is stationary so is $\{Y_t^{(1)}\}$ In the case of linear trend, if we difference again:

$$\begin{aligned} X_t^{(2)} &= X_t^{(1)} - X_{t-1}^{(1)} = (X_t - X_{t-1}) - (X_{t-1} - X_{t-2}) \\ &= (\beta + Y_t - Y_{t-1}) - (\beta + Y_{t-1} - Y_{t-2}) \\ &= Y_t - 2Y_{t-1} + Y_{t-2}, \quad (\equiv Y_t^{(1)} - Y_{t-1}^{(1)} = Y_t^{(2)}), \end{aligned}$$

so that the effect of $\mu_t (= \alpha + \beta t)$ has been completely removed.

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If μ_t is a polynomial of degree $(d-1)$ in t, then dth differences of μ_t will be zero ($d = 2$ for linear trend). Further,

$$X_t^{(d)} = \sum_{k=0}^{d} \binom{d}{k} (-1)^k X_{t-k} = \sum_{k=0}^{d} \binom{d}{k} (-1)^k Y_{t-k}.$$

There are other ways of writing this. Define the difference operator

$$\Delta = (1 - B)$$

where $BX_t = X_{t-1}$ is the *backward shift operator* (sometimes known as the *lag operator* L – especially in econometrics). Then,

$$X_t^{(d)} = \Delta^d X_t = \Delta^d Y_t.$$

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For example, for $d = 2$:

$$\begin{aligned} X_t^{(2)} &= (1-B)^2 X_t = (1-B)(X_t - X_{t-1}) \\ &= (X_t - X_{t-1}) - (X_{t-1} - X_{t-2}) \\ &= (\beta + Y_t - Y_{t-1}) - (\beta + Y_{t-1} - Y_{t-2}) \\ &= (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2}) \\ &= (1-B)^2 Y_t = \Delta^2 Y_t. \end{aligned}$$

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This notation can be incorporated into the ARMA set up; if $\{X_t\}$ is ARMA(p, q),

$$X_t = \phi_{1,p} X_{t-1} + \ldots + \phi_{p,p} X_{t-p} + \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q},$$

$$X_t - \phi_{1,p} X_{t-1} - \ldots - \phi_{p,p} X_{t-p} = \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q}$$

$$(1 - \phi_{1,p} B - \ldots - \phi_{p,p} B^p) X_t = (1 - \theta_{1,q} B - \ldots - \theta_{q,q} B^q)\epsilon_t$$

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That is,

$$\Phi(B)X_t = \Theta(B)\epsilon_t$$

where

$$\begin{aligned}\Phi(B) &= 1 - \phi_{1,p}B - \phi_{2,p}B^2 - \ldots - \phi_{p,p}B^p \\ \Theta(B) &= 1 - \theta_{1,q}B - \theta_{2,q}B^2 - \ldots - \theta_{q,q}B^q\end{aligned}$$

are known as the *associated* or *characteristic polynomials*.

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Further, we can generalize the class of ARMA models to include differencing to account for certain types of non-stationarity, namely,

- X_t is called **ARIMA(p, d, q)** if

$$\Phi(B)(1-B)^d X_t = \Theta(B)\epsilon_t,$$

$$\Phi(B)\Delta^d X_t = \Theta(B)\epsilon_t.$$

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Seasonal adjustment

The model is modified to

$$X_t = s_t + Y_t$$

where

- $\{s_t\}$ is the **seasonal** component,
- $\{Y_t\}$ is zero mean **stationary** process.

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Presuming that the seasonal component maintains a constant pattern over time with period s, there are again several approaches to removing s_t. A popular approach used by Box & Jenkins is to use the operator $(1 - B^s)$:

$$\begin{aligned} X_t^{(s)} &= (1 - B^s)X_t = X_t - X_{t-s} \\ &= (s_t + Y_t) - (s_{t-s} + Y_{t-s}) \\ &= Y_t - Y_{t-s} \end{aligned}$$

since s_t has period s (and so $s_{t-s} = s_t$).