Abstract. We introduce homotopy-algebraic structures on Galois coho-
ology that determine presentations for universal deformation rings of Galois
representations. From this, we deduce presentations for universal deformation
rings of Galois pseudorepresentations. The latter presentation is used to supply
a tangent and obstruction theory for pseudorepresentations. This generalizes
the well-used tangent and obstruction theory for Galois representations. We
also give an application, calculating the ranks of certain Hecke algebras.

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Part 1. Introduction

The theme of this paper is to identify homotopy-algebraic structures on Galois cohomology groups that explain number-theoretic phenomena. More specifically, we describe an $A_{\infty}$-algebra structure on the Galois cohomology of the adjoint representation of a Galois representation $\rho$ and relate it to the deformation theory of $\rho$. We prove that a certain classical hull of this homotopy algebra represents the classical Galois deformation problem. This gives a presentation, in terms of Galois cohomology, of the Galois deformation rings first studied by Mazur [Maz89].

This homotopy algebra may be fairly called a “derived enrichment” of classical moduli of Galois representations. Therefore, there are some relations between this work and, for example, that of Galatius–Venkatesh [GV18] on derived Galois deformation rings; for more comments about this, see §4.4.

However, we do not discuss derived deformation problems here, as our motivation is to identify precise expressions for objects that live squarely in the classical world. Indeed, the original motivation for this homotopy-algebraic study was to find a tangent and obstruction theory for Galois pseudorepresentations. This is done by first finding precise expressions for classical moduli spaces of Galois representations, and then deducing a tangent and obstruction theory for Galois pseudorepresentations using the author’s previous work [WE18].

1. Motivation

1.1. 2-dimensional Galois representations and modular Hecke eigenforms. Consider 2-dimensional representations of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$, with coefficients in a $p$-adic field. The Fontaine–Mazur conjecture [FM95] predicts that certain of them are expected to arise from modular Hecke eigenforms. These two classes of objects have $p$-integral structure; hence they are organized into congruence classes modulo $p$. We label the congruence classes of Galois representations and Hecke eigenforms by $ar{\rho}: G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F})$ and $\bar{f} \in \mathbb{F}[q]$, respectively. Here $\mathbb{F}$ is a finite field of characteristic $p$. Work of Mazur [Maz89, Maz77] introduced the moduli-theoretic study of these congruence classes, putting them in bijection with

- (Galois representations) homomorphisms $R_{\bar{\rho}} \to \overline{\mathbb{Q}}_p$ out of a deformation ring $R_{\bar{\rho}}$ of Galois representations, designed so that commutative $\mathbb{Z}_p$-algebra homomorphisms $R_{\bar{\rho}} \to A$ correspond to strict equivalence classes of Galois representations $\rho_A$ with coefficients in $A$ such that
  (i) $\rho_A : G_{\mathbb{Q}} \to \text{GL}_2(A)$ is congruent to $\bar{\rho}$, and
  (ii) $\rho_A$ satisfies properties expected of Galois representations arising from Hecke eigenforms.
- (Hecke eigenforms) homomorphisms $T \to \overline{\mathbb{Q}}_p$ out of a Hecke algebra $T$, where $T$ is the completion of a Hecke algebra $\mathcal{H}$ (arising from the Hecke action on a finite dimensional $\mathbb{C}$-vector space of modular forms) at a maximal ideal corresponding to $\bar{f}$.

When $\bar{\rho}$ and $\bar{f}$ may be chosen compatibly, which we now assume, it is natural to ask whether there is a local homomorphism $R_{\bar{\rho}} \to T$ arising from the $p$-adic Galois representations attached to Hecke eigenforms. Then, one is led to ask about “$R_{\bar{\rho}} = T$. “
1.2. The irreducible case. When there are no Eisenstein series congruent to \( \bar{f} \), or, equivalently, \( \bar{\rho} \) is absolutely irreducible, it is often possible to prove that \( R_{\bar{\rho}} \sim T \) – this was first carried out by Mazur \[Maz89\] and Wiles \[Wil95\]. Since Wiles’s work, one crucial aspect of this argument is control over \( R_{\bar{\rho}} \) in terms of the arithmetic invariants of Galois cohomology. In the most basic setting, these are

\[
H^1(G, \text{End}_F(\rho)) \text{ and } H^2(G, \text{End}_F(\rho)),
\]

where \( \rho \) may now have arbitrary dimension \( d \). The key control is called a “tangent and obstruction theory:” first-order deformations of \( \rho \), which are represented by homomorphisms

\[
\rho_1 : G \rightarrow \text{GL}_d(F[\varepsilon]/\varepsilon^2) \quad \text{such that } (\rho_1 \mod \varepsilon) = \rho,
\]

are in bijection with the tangent space

\[
H^1(G, \text{End}_F(\rho)) \cong \text{Ext}_F^1(G, \rho, \rho).
\]

And an \( n \)-th order deformation

\[
\rho_n : G \rightarrow \text{GL}_d(F[\varepsilon]/\varepsilon^{n+1}) \quad \text{such that } (\rho_n \mod \varepsilon) = \rho,
\]

induces an element of the obstruction space

\[
H^2(G, \text{End}_F(\rho)) \cong \text{Ext}_F^2(G, \rho, \rho)
\]

that is zero if and only if \( \rho_n \) can be extended to an \( (n+1) \)-st order deformation, i.e.

\[
\rho_{n+1} : G \rightarrow \text{GL}_d(F[\varepsilon]/\varepsilon^{n+2}) \quad \text{such that } (\rho_{n+1} \mod \varepsilon^{n+1}) = \rho_n.
\]

These extensions are a torsor over \( H^1(G, \text{End}_F(\rho)) \).

In practice, one carries out the Taylor–Wiles method of \[Wil95, TW95\] and subsequent developments, which involve auxiliary cases where the tangent and obstruction theory reduces to the simple case \( H^2(\text{aux}, \text{End}_F(\rho)) = 0 \).

1.3. The reducible case. When \( \bar{\rho} \) is reducible, it is necessary to modify the approach above. Let \( d = 2 \) for simplicity. In order to hope for a correspondence with \( T \) in general, we must replace \( R_{\bar{\rho}} \) by a ring \( R_{\text{ps}} \) that parameterizes 2-dimensional pseudorepresentations of Galois groups. So we write \( R_{\text{ps}} \) for clarity. Indeed, isomorphisms \( R_{\text{ps}} \cong T \) have been proven; see e.g. \[BK11, BK15, Deo17, WWE18\].

A 2-dimensional pseudorepresentation of \( G_\mathbb{Q} \) valued in \( A \), written \( D_A : G \rightarrow A \), amounts to a pair of functions

\[
D_A = \{ \text{Tr}, \det : G \rightarrow A \}
\]

obeying properties expected of such functions arising from characteristic polynomials of a 2-dimensional representation. For a precise definition of a \( d \)-dimensional pseudorepresentation due to Chenevier \[Che14\], see \[10.2\]. Given a representation \( \rho \), we write \( \psi(\rho) \) for the induced pseudorepresentation.

For the moment, we take \( \bar{D} = \psi(\bar{\rho}) : G \rightarrow \bar{F} \) to be the pseudorepresentation given by \( \{ \text{Tr} \bar{\rho}, \det \bar{\rho} \} \). Then we let \( R_{\text{ps}} \) be the “universal pseudodeformation ring” for \( \bar{D} \); it has the universal property that local \( \mathbb{Z}_p \)-algebra homomorphisms \( R_{\text{ps}} \rightarrow A \) are in bijection with pseudorepresentations \( D_A : G \rightarrow A \) such that

(i) \( D_A \) is congruent to \( \bar{D} \), i.e. the composite of \( D_A \) and \( A \rightarrow A/m_A \cong F \) is equal to \( \bar{D} \); and
(ii) $D_A$ satisfies properties of Galois representations arising from Hecke eigenforms; such conditions are translated from representations to pseudorepresentations (of any dimension) in the author’s work with Preston Wake [WWE17a].

However, a tangent and obstruction theory for pseudorepresentations has been lacking. For example, Thorne remarks that “the ring $R^\rho$ is difficult to control using Galois cohomology, a tool which is essential in other arguments” [Tho15, pg. 786]. To this author’s knowledge, no formulation of obstruction theory has been produced. There is a partial characterization of and canonical filtration on the tangent space due to Bellaïche [Bel12], following on his work with Chenevier [BC09], in the case where the semi-simple $\bar{\rho}$ inducing $\bar{D}$ has distinct simple factors. However, the tangent space is only characterized when there are two simple factors [Bel12, Thm. A].

1.4. Goals of the paper. The main goal of this paper is to procure a tangent and obstruction theory for pseudorepresentations, so that the relationships between $R^\rho$, $T_c$, and arithmetic might be better understood. Indeed, the point of the tangent and obstruction theory is to elucidate how $R^\rho$ is related to arithmetic invariants. This is novel even for 2-dimensional representations, but the result is in general dimension and under a mild hypothesis.

Along the way, we must cope with the fact that pseudorepresentations are “non-linear,” in contrast with the linear structure of representations (they form an abelian category). In particular, there is no “$\text{Ext}^i$” for pseudorepresentations. In this sense, what we produce is not a conventional tangent and obstruction theory; perhaps it does not deserve this name. Instead, we write down a presentation for Galois pseudodeformation rings in terms of a homotopy algebra structure on $H^\bullet(G, \text{End}_F(\rho)) \cong \text{Ext}^\bullet_F(G, \rho, \rho)$ that we referred to at the outset. We will justify the perspective that such homotopy-algebraic structures are unavoidable even if one merely wants to understand the tangent space of $R^\rho$.

This presentation is computed as follows:

1. Present moduli spaces of Galois representations, including but not limited to the deformation spaces Spec $R_\rho$ discussed above, in terms of homotopy products on $H^\bullet(G, \text{End}_F(\rho))$.
2. Apply the author’s previous work [WE18], which explains the relationship between Galois pseudodeformation rings and moduli spaces of Galois representations, in order to deduce a presentation for $R^\rho$.

As the presentations of step (1) are novel in number theory (but familiar in non-commutative geometry; see §4.5), explaining them is a natural secondary goal of this paper. Our point of departure is to articulate the homotopy algebra structure — which is an $A_\infty$-structure — on $H^\bullet(G, \text{End}_F(\rho))$.

Once these goals are achieved, we also

3. demonstrate that the theory may be applied to the deformation theory of Galois representations with an auxiliary condition imposed, and
4. as an application, compute the ranks of certain $p$-adic modular Hecke algebras.

2. Background on $A_\infty$-algebras

In preparation to state the main results, and as this article is intended in part to introduce these particular homotopy-algebraic notions to number theorists, we
introduce the required background on $A_{\infty}$-algebras. We follow [LV12 Ch. 9]. See §5 for a fuller introduction.

2.1. Definition via the bar construction. Let $\hat{T}_F(V)$ denote the complete free (associative) $F$-algebra on a $F$-vector space $V$. Let $\hat{S}_F(V)$ denote the complete free commutative $F$-algebra on $V$. A $F$-basis $\{v_1, \ldots, v_n\}$ for $V$ determines an isomorphism $\hat{S}_F(V) \cong F[[v_1, \ldots, v_n]]$. That is, $\hat{S}_F(V) \cong \prod_{n \geq 0} \text{Sym}_F^n V$; similarly, $\hat{T}_F(V) \cong \prod_{n \geq 0} V^\otimes n$.

When $H = H^*$ is a $\mathbb{Z}$-graded $F$-vector space, an $A_{\infty}$-algebra structure is a sequence $m = (m_n)_{n \geq 1}$ of $F$-linear maps

$$m_n : H^\otimes n \to H \quad \text{of degree } 2 - n,$$

for $n \in \mathbb{Z}_{\geq 1}$, satisfying many compatibility relations (see [5.1] for this and further details). For the moment, we note that when $m_n = 0$ for $n \geq 3$, the compatibility relations for $m_1, m_2$ are exactly the axioms of a differential graded algebra with differential $m_1$ and multiplication $m_2$.

We introduce morphisms and quasi-isomorphisms of $A_{\infty}$-algebras. A morphism of $A_{\infty}$-algebras $f : H \to H'$ is a sequence $f = (f_n)_{n \geq 1}$ of maps

$$f_n : H^\otimes n \to H' \quad \text{of degree } 1 - n,$$

for $n \in \mathbb{Z}_{\geq 1}$, satisfying certain relations that will be explained later. One of the relations is that $f_1$ is a morphism of complexes $(H, m_1) \to (H', m'_1)$. We call $f$ a quasi-isomorphism when $f_1$ is a quasi-isomorphism of complexes.

Both the definition of an $A_{\infty}$-algebra and the main result will become clearer by reformulating the notion of an $A_{\infty}$-algebra through a dualized version of the bar construction, which we now explain in a stepwise way. First we suspend the maps $m_n$, i.e. we suspend $H$ using notation $\Sigma$ (alternatively written as [1]) so that each $m_n$ induces a $F$-linear map

$$\Sigma m_n : \Sigma H^\otimes n := (\Sigma H)^{\otimes n} \to \Sigma H \quad \text{of degree } 1.$$

Then, presuming that $H^i$ is finite-dimensional for all $i \in \mathbb{Z}$, we take the linear duals of the composite of (2.1.3) with the projection $\Sigma H \to \Sigma H^i$. Take the sum of these dual maps over their domains, yielding

$$m_n^* : \Sigma H^* := \bigoplus_{i \in \mathbb{Z}} (\Sigma H^i)^* \to (\Sigma H^*)^{\otimes n}, \text{ also of degree } 1.$$

Next we take the product over the codomains as $n$ varies, writing

$$m^* : \Sigma H^* \to \prod_{n \geq 1} (\Sigma H^*)^{\otimes n}.$$

Finally, we extend the domain of $m^*$ to the complete free associative algebra $\hat{T}_F \Sigma H^*$ via the Leibniz rule, producing

$$m^* : \hat{T}_F \Sigma H^* \to \hat{T}_F \Sigma H^* \quad \text{of degree } 1.$$

Altogether, we write

$$\text{Bar}^*(H) = \text{Bar}^*(H, m) := (\hat{T}_F \Sigma H^*, m^*, \pi)$$

for this complete free graded $F$-algebra with derivation $m^*$, where $\pi$ denotes the standard multiplication operation.
Analogously, a sequence of maps \( f = (f_n)_{n \geq 1} \) as in (2.1.2) induces a homomorphism of complete free \( \mathbb{F} \)-algebras
\[
\text{Bar}^*(f) : \text{Bar}^*(H') \rightarrow \text{Bar}^*(H).
\]

Notice that nothing we have said so far depends on the relations on \( m = (m_n)_{n \geq 1} \) defining an \( A_\infty \)-algebra or the relations on \( f \) defining a morphism of \( A_\infty \)-algebras. The following statement contains a concise and equivalent formulation of these relations.

**Fact 2.1.4 (Bar construction).** Assume that \( H^i \) is finite-dimensional for all \( i \in \mathbb{Z} \). A sequence of maps \( m = (m_n)_{n \geq 1} \) as in (2.1.1) make \( (H, m) \) an \( A_\infty \)-algebra if and only if \( \text{Bar}^*(H, m) \) is a dg-algebra. That is, \( m \) defines an \( A_\infty \)-structure on \( H \) if and only if \( m^* \) is a differential, i.e. \( (m^*)^2 = 0 \).

Likewise, a sequence of maps \( f = (f_n)_{n \geq 1} \) as in (2.1.2) make \( f : H \rightarrow H' \) a morphism of \( A_\infty \)-algebras if and only if the \( \mathbb{F} \)-algebra homomorphism \( \text{Bar}^*(f) \) is a homomorphism of dg-algebras, i.e. \( \text{Bar}^*(f) \circ m^* = m^* \circ \text{Bar}^*(f) \). Moreover,

1. One can drop the condition that each \( H^i \) is finite-dimensional and produce a co-complete co-free co-dg-algebra (see Definition 5.4.1).
2. This construction induces an isomorphism of categories between \( A_\infty \)-algebras and co-complete co-free co-dg-algebras.

**Proof.** By direct computation. See e.g. [LV12, Lem. 9.2.2 and §9.2.11].

2.2. **A theorem of Kadeishvili.** The remainder of the background from homotopical algebra that we require originates in a theorem of Kadeishvili [Kad82].

**Fact 2.2.1 (Kadeishvili).** Let \((C, d_C, m_{2,C})\) be a dg-\( \mathbb{F} \)-algebra. Let \( H = H^*(C) \) be the graded \( \mathbb{F} \)-vector space of cohomology of the complex \((C, d_C)\).

There is an \( A_\infty \)-algebra structure \( m = (m_n)_{n \geq 1} \) on \( H \) and a quasi-isomorphism of \( A_\infty \)-algebras \( f = (f_n)_{n \geq 1} : (H, m) \rightarrow (C, (m_{n,C})_{n \geq 1}) \) (where we let \( m_{1,C} = d_C \) and \( m_{n,C} = 0 \) for \( n \geq 3 \)) such that

1. \( m_1 = 0 \)
2. \( m_2 = (m_{2,C} \mod \text{image}(d_C)) \)
3. \( \text{pr} \circ f_1 = \text{id}_H \), where \( \text{pr} \) is the projection from \( \ker(d_C) \) to \( H \).

These structures are unique up to non-unique isomorphism.

**Proof.** See e.g. [LV12, Cor. 9.4.8]; see also §5.2 for more details.

The idea is that the standard graded algebra structure \((H, m_2)\) on the cohomology \( H \) of a dg-algebra \( C \) can be enriched into the structure of an \( A_\infty \)-algebra \((H, m)\) that does not lose information from \( C \). In §5.2, we explain that the choice of a homotopy retract between \( H \) and \( C \) induces a particular choice of \( f \) and \( m \).

**Remark 2.2.2.** Another way of expressing Fact 2.2.1 is that the higher \( A_\infty \)-products \( m_n, n \geq 3 \) encode the information lost by passing to the cohomology algebra. When no information is lost, i.e. there is a choice of \( m \) in the statement such that \( m_n = 0 \) for \( n \geq 3 \), the dg-algebra is called formal. The formal case makes for relatively straightforward deformation theory (see Remark 3.1.4).

By way of giving an example of this fact, we introduce the Galois cohomology objects that appear in the main theorem, starting with the standard construction.
of Galois cohomology. Let $V$ be a $\mathbb{F}[G]$-module. Let

$$C^*(G, V) \cong \bigoplus_{i \geq 0} C^i(G, V)$$

denote the complex of inhomogeneous continuous cochains on the profinite group $G$, graded by degree $i$. When $V$ has the structure of a $\mathbb{F}$-algebra, one may check (see e.g. [NSW08, Prop. 1.4.1]) that the composition $m_{2,C}$ of the standard cup product of cochains with the multiplication map $V \otimes_{\mathbb{F}} V \to V$, namely,

$$C^i(G, V) \otimes_{\mathbb{F}} C^j(G, V) \to C^{i+j}(G, V \otimes_{\mathbb{F}} V) \to C^{i+j}(G, V),$$

makes $C^*(G, V)$ a dg-$\mathbb{F}$-algebra. That is, the Leibniz rule is satisfied.

We write $H^\bullet(G, V)$ for the graded $\mathbb{F}$-vector space of cohomology of $C^\bullet(G, V)$. The dg-algebra structure on $C^\bullet(G, V)$ induces a graded algebra multiplication $m_2 : H^\bullet(G, V) \otimes_{\mathbb{F}} H^\bullet(G, V) \to H^\bullet(G, V)$ on $H^\bullet(G, V)$. Now we may apply Fact 2.2.1, producing an $A_\infty$-algebra structure $m = (m_n)_{n \geq 1}$ on $H^\bullet(G, V)$ extends the native dg-algebra structure on the graded algebra $(H^\bullet(G, V), 0, m)$. That is, it extends the usual cup product in cohomology $m_1 = 0$, $m_2 = (m_{2,C} \mod \text{image}(d^{C^i}))$.

For the purposes of this introduction, our case of interest is where $V = \text{End}_{\mathbb{F}}(\rho)$, where $\rho$ is absolutely irreducible as in §1.2 above.

### 2.3. The classical hull

Finally, we define the classical hull of a dg-algebra.

**Definition 2.3.1.** The classical hull $A(C) = A(C, d, m)$ of a dg-algebra $(C, d, m_{2,C})$ is the ring $A(C) := C^0 / \text{d}(C^{-1})$ concentrated in graded degree zero, taken as a (classical) ring. This functor is left adjoint to the functor sending classical (associative) algebras $(A, m)$ to the dg-algebra $(A[0], 0, m[0])$ concentrated in degree zero as $A$ and with the zero differential. That is,

$$\text{Hom}_{\text{dg-}\mathbb{F}}(C, A[0]) = \text{Hom}_{\mathbb{F}}(A(C), A).$$

Since we are most interested in the classical hull of the dg-algebra $(\hat{T}_{\mathbb{F}} \Sigma H^\bullet, m^*, \pi)$ produced by the bar construction in Fact 2.1.4 we describe this case in particular.

**Example 2.3.2.** The classical hull of the dg-algebra $\text{Bar}^*(H) = (\hat{T}_{\mathbb{F}} \Sigma H^*, m^*, \pi)$ produced by the dualized bar construction of Fact 2.1.4 is

$$\frac{\hat{T}_{\mathbb{F}}(\Sigma H^1)^*}{(m^*((\Sigma H^2)^*))}.$$

Indeed, notice that any map $\hat{T}_{\mathbb{F}} \Sigma H^* \to A$ factors through $\hat{T}_{\mathbb{F}}(\Sigma H^1)^* \to A$, as $(\Sigma H^1)^*$ is the degree zero part of $\Sigma H^*$. Then, calculate using the Leibniz rule that the ideal generated by the projection of $m^*(\Sigma H^*)$ to $\hat{T}_{\mathbb{F}}(\Sigma H^1)^*$ is $(m^*((\Sigma H^2)^*))$.

### 3. Main results

We present results toward each of the Goals (1)-(4) of §1.4. In particular, we address deformations of pseudorepresentations in §3.3.
3.1. Results, Part I: determination of moduli spaces of representations.
As in §1.3, let \( \rho : G \to \text{GL}_d(F) \) be a semi-simple representation with simple summands \( \rho_i : G \to \text{GL}_{d_i}(F) \), \( 1 \leq i \leq r \). We write
\[
\rho \cong \bigoplus_{i=1}^{r} \rho_i,
\]
thinking of this as a block diagonal decomposition of the homomorphism \( \rho \). We impose the running assumption that \( \rho_i \not\cong \rho_j \) for \( i \neq j \) (the multiplicity-free condition), and make a further mild restriction explained in Definition 10.1.1. We call such \( \rho \) a multiplicity-free residual semi-simplification, as we will study the moduli of representations that deform some \( F \)-valued representation whose semi-simplification is \( \rho \). This moduli space, called \( \text{Rep}_\rho \), is set up in §10.

Theorem [11.3.1] gives a presentation of \( \text{Rep}_\rho \) in terms of \( A_\infty \)-algebra structure on \( H^\bullet(G, \text{End}_F(\rho)) \). In this introduction we present the simplest case, where \( \rho \) is irreducible, i.e. \( r = 1 \). In this case \( \text{Rep}_\rho \cong \text{Spec } R_\rho \), where \( R_\rho \) is the usual deformation ring (see §1.2).

**Theorem 3.1.1** (Special case \( r = 1 \) of Theorem [11.3.1]). Let \( \rho \) be an absolutely irreducible \( F \)-valued representation of \( G \). Assume that \( H^i(G, \text{End}_F(\rho)) \) is finite-dimensional for all \( i \geq 0 \). As described in Fact 2.2.1, there exists a structure of \( A_\infty \)-algebra \( m = (m_n)_{n \geq 1} \) on \( H = H^\bullet(G, \text{End}_F(\rho)) \) that is compatible with the \( \text{dg} \)-algebra \( C = C^\bullet(G, \text{End}_F(\rho)) \) in that there exists quasi-isomorphism of \( A_\infty \)-algebras
\[
f : (H, m) \to (C, d_C, m_{2,C})
\]

Let \( (\bar{T}_F \Sigma H^\bullet(G, \text{End}_F(\rho))^*, m^*, \pi) \) denote the complete \( \text{dg} \)-algebra arising from the bar construction on \( (H^\bullet(G, \text{End}_F(\rho)), (m_n)_{n \geq 1}) \).

These data determine a \( \text{dg} \)-algebra homomorphism
\[
\bar{T}_F \Sigma H^1(G, \text{End}_F(\rho))^* \to R_\rho[0],
\]
which factors through an isomorphism of classical complete commutative algebras
\[
\frac{\bar{T}_F \Sigma H^1(G, \text{End}_F(\rho))^*}{(m^*(\Sigma H^2(G, \text{End}_F(\rho))^*))} \cong R_\rho,
\]
from the abelianization of the classical hull of \( \bar{T}_F \Sigma H^*(H, m) \) to \( R_\rho \).

Conventional tangent and obstruction theory is an immediate corollary. Let \( h^i \) denote the \( F \)-dimension of \( H^i(G, \text{End}_F(\rho)) \).

**Corollary 3.1.2** (Tangent and obstruction theory). There is a tangent and obstruction theory for deformations of \( \rho \), as outlined in §1.2. Moreover, there is a bound on Krull dimension
\[
h^1 - h^2 \leq \dim(R_\rho) \leq h^1.
\]

Indeed, the bound on Krull dimension can be read off the presentation of Theorem 3.1.1. One can also derive a class in \( H^2(G, \text{End}_F(\rho)) \) from the composition of the presentation of \( R_\rho \) with a homomorphism \( R_\rho \to F[\varepsilon]/\varepsilon^{n+1} \) corresponding to \( \rho_n \). This is equal to a class defined directly through representation theory.

Here are some remarks that give further perspective on the formula.
Remark 3.1.3. It is implicit in the theorem statement that there is no canonical choice of $A_{\infty}$-structure on $H^*(G, \text{End}_{\mathbb{F}}(\rho))$. Indeed, the tangent theory of $\mathcal{M}_2$ amounts to a canonical surjection of $\hat{S}_2 \Sigma H^1(G, \text{End}_{\mathbb{F}}(\rho))^*$ onto $R_\rho / m_\rho^2$, but one may readily calculate by hand to observe that it has no canonical lift to $R_\rho / m_\rho^3$.

Remark 3.1.4. Given the surjection $\hat{S}_2 \Sigma H^1(G, \text{End}_{\mathbb{F}}(\rho))^* \twoheadrightarrow R_\rho$, we see that the presentation for $R_\rho / m_\rho^3$ is determined by the projection of $m^*$ to $\Sigma H^2(G, \text{End}_{\mathbb{F}}(\rho))^* \rightarrow \text{Sym}_2^\Sigma(\Sigma H^1(G, \text{End}_{\mathbb{F}}(\rho))^*)$.

In fact, this is the $\mathbb{F}$-linear dual of the cup product map $m_2$. This cup product map is canonical, i.e. it does not depend on the choice of $A_{\infty}$-structure. The fact that a cup product obstructs the extension of first-order deformations to second-order deformations is well-known. See e.g. [Max89 §1.6, Remark; pg. 400].

This observation has been applied to great effect when $m^*$ vanishes in degree greater than 2, which is the “formal” case. See §4.6 and, e.g., [GM88].

Remark 3.1.5. An abelianization appears in Theorem 3.1.1. In fact, the theorem follows from a more general theorem, Corollary 7.4.5 which is the main theorem of Part 2, and applies to representations of an associative $\mathbb{F}$-algebra $E$. It is a new result in non-commutative deformation theory; see §4.5 for more comments on this. Corollary 7.4.5 specializes to the case $r = 1$ in Corollary 6.2.6 and yields Theorem 3.1.1 upon applying it to $E = \mathbb{F}[G]$.

3.2. Geometric invariant theory. This section prepares notation for the statement of Theorem 3.3.1, which provides presentations for pseudodeformation rings. This notation express invariant subrings that arise from geometric invariant theory (GIT). We will suppress the invariant theory in favor of combinatorial expressions. Further details about GIT are found in §10.3.

Let the pseudorepresentation $D : G \rightarrow \mathbb{F}$ arise from a semi-simple representation $\rho : G \rightarrow \text{GL}_d(\mathbb{F})$ with distinct absolutely irreducible summands

$$\rho \simeq \bigoplus_{i=1}^{r} \rho_i.$$  

Notation 3.2.1 ([Bel12 §2.2]). We set up the following combinatorial objects on the integers from 1 to $r$.

- Write $r$ for the set $\{1, \ldots, r\}$.
- A path of length $l$ is a function $\gamma : \{0, \ldots, l\} \rightarrow \{1, \ldots, r\}$. We write $l = l_\gamma$ for the length of $\gamma$.
- We say that a path $\gamma$ goes from $i$ to $j$ when $\gamma(0) = i$ and $\gamma(l_\gamma) = j$.
- We call $\gamma$ closed if $\gamma(0) = \gamma(l_\gamma)$. In this case, we may consider the domain of $\gamma$ to be $\mathbb{Z}/l_\gamma \mathbb{Z}$.
- We call a path $\gamma$ simple if

  $$\gamma(i) = \gamma(j) \text{ and } i \neq j \implies \{i, j\} = \{0, l_\gamma\},$$

  That is, a path is simple if it is injective, or it is closed and maximally injective.
- A cycle is an equivalence class of closed paths under the equivalence relation $\gamma \sim \gamma'$ defined by

  $$\gamma \sim \gamma' \iff \left\{ \begin{array}{l} l_\gamma = l_{\gamma'} =: l, \text{ and } \\
  \exists k \in \mathbb{Z}/l_\gamma \mathbb{Z} \text{ such that } \gamma(i) = \gamma'(i + k) \forall i \in \mathbb{Z}/l_\gamma \mathbb{Z}. \end{array} \right.$$
A cycle is called simple if one (equivalently, all) of its constituent closed paths is simple.

Write \( SC(l) \) for the set of simple cycles in \( r \) of length \( l \), and write \( SC(r) \) for the set of all simple cycles in \( r \) (of any length).

For \( i, j \in r \), write \( SCC(i, j) \) for the paths \( \gamma \) from \( i \) to \( j \) such that the concatenation of \( \gamma \) with the length 1 path from \( j \) to \( i \) is a simple closed cycle. In particular, \( i = j \) is allowed, but \( SCC(i, i) = \emptyset \) in this case. ("SCC" stands for "simple closed complements.")

Given a representative \( \gamma \) of an element of \( SC(l) \),

In the following notation, we express group cohomology as Ext-groups for convenience. These expressions are all contained in the canonical isomorphism

\[
\text{Ext}_{F[G]}^*(\rho, \rho) \cong H^*(G, \text{End}_F(\rho)).
\]

For more on this, see \[\text{11.1}\]

**Notation 3.2.2.** The following objects enrich the foregoing notation from finite sets to vector spaces coming from Ext.

- For \( i, j \in r \), let
  \[
  \text{Ext}^k_{G}(i, j) := \text{Ext}^k_{F[G]}(\rho_i, \rho_j) \cong H^k(G, \rho_j \otimes_F \rho_i^*)..
  \]
- Given a path \( \gamma \) on \( r \), we write
  \[
  \Sigma \text{Ext}^1_G(\gamma)^* := \bigotimes_{i=0}^{l_{\gamma}-1} \Sigma \text{Ext}^1_G(\gamma(i), \gamma(i+1))^*.
  \]
- Let \( C(D) \) be the directed graph whose vertices \( \{\rho_i\}_{i=1}^r \) and whose arrows from \( \rho_i \) to \( \rho_j \) are a choice of basis for \( \text{Ext}^1_G(\rho_j, \rho_i) \). We will refer to the property of being strongly connected, i.e. the existence of a directed path between any two vertices, as well as the decomposition into strongly connected components.
- Let \( h^1_{ij} := \dim_F \text{Ext}^1_G(\rho_j, \rho_i) \).
- Let \( H_1(C(D)) \) be the simplicial homology of the simplicial 1-complex naturally arising from \( C(D) \). Let \( \mathbb{N}(SC(C(D))) \) be the free commutative monoid on simple cycles \( C(D) \). Let \( J \) be the kernel of its natural map to \( H_1(C(D)) \). That is, \( J \) consists of \( \mathbb{N} \)-linear combinations of simple cycles that have the same underlying sets-with-multiplicity of arrows. Finally, let \( h^2(C(D)) \) be the set
  \[
  J \sim (J \cdot (\mathbb{N}(SC(C(D))) \setminus \{0\}))
  \]
  and let \( H^2(C(D)) \) be the \( F \)-vector space with basis \( h^2(C(D)) \).

**Remark 3.2.3.** When we use the notation \( \text{Ext}^1_G(\gamma) \) for \( \gamma \in SC(r) \), we will work with symmetric tensors. Then the choice of representative path of a cycle does not matter.

Finally, we define an invariant subring that will appear often in our presentations for pseudodeformation rings.

**Definition 3.2.4.** Let \( R^1_{\theta} \) denote the local ring that is the image of

\[
\hat{S}_F \bigoplus_{\gamma \in SC(r)} \Sigma \text{Ext}^1_G(\gamma)^* \longrightarrow \hat{S}_F \bigoplus_{i, j \in r} \Sigma \text{Ext}^1_G(i, j)^* \cong \hat{S}_F \Sigma \text{Ext}_{F[G]}^*(\rho, \rho)^*.
\]

We will supply references for the following fundamental facts of GIT in \[\text{11.6}\]
Fact 3.2.6. \( R_1^D \) is reduced, normal, and Cohen-Macaulay. If \( C(D) \) decomposes into strongly connected components \( \coprod C(D_a) \) where \( D = \bigoplus_a D_a \), then there is a canonical isomorphism

\[
R_1^D \cong \bigotimes_a R_1^{D_a}.
\]

When \( C(D) \) is strongly connected, then its Krull dimension is

\[
\dim R_1^D = 1 - r + \sum_{1 \leq i,j \leq r} h_{ij}^1.
\]

We substantiate this basic fact in §11.7.

Fact 3.2.7. Let \( K \) denote the kernel of \((3.2.5)\) and let \( m \) the maximal ideal of the codomain. There exists a canonical isomorphism

\[
H_2(C(D))^* \cong K/mK.
\]

Remark 3.2.8. The construction \( C(D) \) is a quiver, giving us access to the extensive literature studying representations of quivers and of quivers with relations. For more on this, see §11.6.

3.3. Results, Part II: determination of moduli spaces of pseudorepresentations. Let \( D = \psi(\rho) : G \to F \) be the \( d \)-dimensional pseudorepresentation induced by \( \rho \), as in §1.3. Here is the main theorem.

Theorem 3.3.1. Let \( D := \psi(\rho) \), where \( \rho \) has \( r \) distinct absolutely irreducible factors \( \rho \simeq \bigoplus_{i=1}^r \rho_i \). Assume that \( H^i(G, \text{End}_F(\rho)) \) is finite-dimensional for all \( i \geq 0 \).

Choose a structure of \( A_{\infty}\)-algebra \( m = (m_n)_{n \geq 1} \) on \( H^* (G, \text{End}_F(\rho)) \) and a quasi-isomorphism to the dg-algebra \( (C^* (G, \text{End}_F(\rho)), d_C, m_{2,C}) \), as described in Fact 2.2.1, satisfying the condition of compatibility with the decomposition

\[
\text{End}_F(\rho) \cong \bigoplus_{i,j \in r} \text{Hom}_F(\rho_i, \rho_j)
\]

explained in Example 7.2.3.

The choices above induce an isomorphism

\[
(3.3.2) \quad \frac{\bigoplus m^* \Sigma \text{Ext}_G^2(\rho_j, \rho_i)^* \otimes \left( \bigoplus_{\gamma \in \text{SCC}(i,j)} \Sigma \text{Ext}_G^1(\gamma)^* \right)}{R_1^D} \cong R_D
\]

Remark 3.3.3. Because the formula for \( R_D \) is rather complex, we supply the following intuitive interpretation.

Generators are cycles. Only cycles \( \text{Ext}_G^1(\gamma)^* \) of tensors of \( \Sigma \text{Ext}_G^1(\rho_j, \rho_i)^* \) \((i, j \in r)\) will be detected by pseudorepresentations. Indeed, a choice of extension class

\[
e = (e_{i,j}) \in \text{Ext}_G^1(\rho, \rho) \cong \bigoplus_{i,j \in r} \text{Ext}_G^1(\rho_j, \rho_i)
\]

defines a first order deformation of the form (we take \( r = 3 \) for concreteness)

\[
\rho_e := \rho + \varepsilon e \cong \begin{pmatrix}
\rho_1 + \varepsilon e_{11} & \varepsilon e_{12} & \varepsilon e_{13} \\
\varepsilon e_{21} & \rho_2 + \varepsilon e_{22} & \varepsilon e_{23} \\
\varepsilon e_{31} & \varepsilon e_{32} & \rho_3 + \varepsilon e_{33}
\end{pmatrix}
\]
and only products of the elements $e_{ij}$ over a cycle will appear in the diagonal, and thereby be detectable by the trace. The simple cycles generate the monoid of cycles, and $R^+_D$ is generated by these cycles.

**Relations are obstructed sub-paths of cycles.** However, not every cycle has factors that can multiply together and still form a homomorphism that is detectable by a central function. The obstructions to the appearance of a cycle represented by $\gamma$ consist precisely of elements of $\text{Ext}^2_G(\rho_j, \rho_i)$ that are the image of $m_n$ on any sub-path $\gamma'$ from $j$ to $i$ of the cycle $\gamma$, that is

$$m_n : \text{Ext}^1_G(\rho_j, \rho_{\gamma(1)}) \otimes \cdots \otimes \text{Ext}^1_G(\rho_{\gamma(n-1)}, \rho_i) \rightarrow \text{Ext}^2_G(\rho_j, \rho_i).$$

This is the $m^* \Sigma \text{Ext}^2_G(...)$-factor of the denominator of (3.3.2). The rest of the denominator accounts for the complement of $\gamma'$ in $\gamma$; that is, we must complete the obstructed path $\gamma'$ to the cycle $\gamma$ to calculate its influence on pseudorepresentations.

**Remark 3.3.4.** This expression for $R_D$ decomposes into the strongly connected components of the directed graph $\mathcal{C}(D)$ of Notation 3.2.2. In that notation, we have

$$R_D \cong \bigotimes_a R_{D_a}.$$ 

This is consonant with the fact that each cycle is supported on exactly one strongly connected component. Because this decomposition does not simplify the formulas, we do not use it in the expression of Theorem 3.3.1 and its corollaries.

A tangent and obstruction theory can be derived from Theorem 3.3.1. First we discuss the tangent space. Let

$$t_D := (m_D/m^2_D)^*$$

denote the tangent space of $R_D$.

**Corollary 3.3.5 (The tangent space).** The tangent space $t_D$ is isomorphic to

$$\bigoplus_{\gamma \in \mathcal{S}(r)} \ker \left( \Sigma \text{Ext}^1_G(\gamma) \rightarrow \bigoplus_{0 \leq i < j \leq l_\gamma} \Sigma \text{Ext}^2_G(\rho_{\gamma(j)}, \rho_{\gamma(i)}) \otimes \Sigma \text{Ext}^1_G(\gamma') \right)$$

where $\gamma'$ is the complementary path in $\gamma$ to the subpath $\gamma(i), \gamma(i+1), \ldots, \gamma(j)$ of $\gamma$, and the map parameterized by $\gamma$, $i, j$ sends

$$e(\gamma) := e_0 \otimes \cdots \otimes e_i \mapsto m_{j-i+1}(e_i \otimes \cdots \otimes e_j) \otimes e(\gamma').$$

Here $e(\gamma')$ denotes the tensor factors of $e(\gamma)$ indexed by $\gamma'$.

**Warning 3.3.6.** It is very important to keep in mind that the various direct sums in the statement of Theorem 3.3.1 and Corollary 3.3.5 are non-canonical. Instead, there is a canonical filtration as follows, whose graded pieces can be found among the summands in Corollary 3.3.5.

There is a canonical filtration of the tangent space, the “complexity filtration” of Bellaïche [Bel12, §3]. This is an increasing filtration, where lower complexity degree corresponds to greater reducibility (or less irreducible), in the sense of ideals of reducibility defined in Bellaïche–Chenevier [BC09, §1.5.1] and the derivative notion of complexity of a pseudodeformation of [Bel12, §2.4]. Correspondingly, a lower bound on reducibility (equivalently, an upper bound on complexity) produces a closed condition in $\text{Spec } R_D$. 
In the present terms, the complexity degree it is precisely the number of tensor factors, i.e. the length of \( \gamma \). So we may index it from 0 to \( r \) as
\[
0 = \text{Fil}_0 t_D \subset \text{Fil}_1 t_D \subset \cdots \subset \text{Fil}_r t_D \subset \text{Fil}_r t_D = t_D.
\]

**Corollary 3.3.7.** The \( k \)-th graded factor of the complexity filtration on \( t_D \) are canonically isomorphic to the summand of the expression in Corollary 3.3.5 labeled by \( \gamma \in SC(r) \) such that \( l_\gamma = k \). That is, there is a canonical isomorphism
\[
\frac{\text{Fil}_k t_D}{\text{Fil}_{k-1} t_D} \cong \bigoplus_{\substack{\gamma \in SC(r) \\ l_\gamma = k}} \ker \left( \Sigma \text{Ext}^1_G(\gamma) \to \bigoplus_{0 \leq i < j \leq k} \Sigma \text{Ext}^2_G(\rho_{\gamma(j)}, \rho_{\gamma(i)}) \otimes \Sigma \text{Ext}^1_G(\gamma') \right)
\]

**Remark 3.3.8.** Implicit in the statement is the fact that the kernel does not depend on the choice of \( A_\infty \)-structure on \( \text{Ext}^*_G(\rho, \rho) \).

**Remark 3.3.9.** This refines the main theorem of [Bel12]: [Thm. 1, loc. cit.] states that \( \text{Fil}_k t_D / \text{Fil}_{k-1} t_D \) injects into a sum of kernels that is similar to the expression above, but lacks the \( A_\infty \)-products \( m_n \) for \( n \geq 3 \). That is, only the terms arising from cup products are used in loc. cit.

We derive bounds on the tangent dimension of \( R_D \) from its presentation. Let
\[
h^1_{ij} := \dim \text{Ext}^1_G(\rho_j, \rho_i),
\]
and let
\[
h^1(\gamma) := \prod_{0 \leq i < j} h^1_{\gamma(i), \gamma(i+1)} = \dim \text{Ext}^1_G(\gamma).
\]

**Corollary 3.3.10 (Tangent dimension).** The dimension of \( t_D \) satisfies
\[
\sum_{\gamma \in SC(r)} \left( h^1(\gamma) - \sum_{0 \leq i < j \leq l_\gamma} h^2_{ij} \cdot h^1(\gamma') \right) \leq \dim \text{t}_D \leq \sum_{\gamma \in SC(r)} h^1(\gamma)
\]
where \( \gamma' = \gamma'(\gamma, i, j) \) as in Corollary 3.3.5.

Next we present an obstruction theory.

**Corollary 3.3.11 (Obstruction theory).** Via the presentation of \( R_D \) in Theorem 3.1.4 there is associated to an \( n \)-th order pseudodeformation \( D_n : G \to F[\varepsilon]/\varepsilon^{n+1} \) of \( D \)
\begin{enumerate}
  \item an element of \( \alpha(D_n) \in H_2(C(D)) \) arising from the map \( R^1_D \to F[\varepsilon]/\varepsilon^{n+1} \) associated to \( D_n \).
  \item If \( \alpha(D_n) = 0 \), there is an element \( \beta(D_n) \) in
  \[
  \bigoplus_{i,j \in r} \text{Ext}^2_G(\rho_j, \rho_i) \otimes \bigoplus_{\gamma \in SC(i,j)} \text{Ext}^1_G(\gamma)
  \]
  associated to \( D_n \).
\end{enumerate}
Moreover, \( \alpha(D_n) \) and \( \beta(D_n) \) vanish if and only if \( D_n \) extends to an \((n+1)\)-st order pseudodeformation.

**Remark 3.3.12.** There exists some \( n_0 \in \mathbb{Z}_{\geq 1} \) dependent only on \( r \) and the \( h^1_{ij} \) such that for \( n \geq n_0 \), \( \alpha(D_n) \) vanishes.
When $H^2(G, \text{End}_F(\rho)) \cong \text{Ext}^2_G(\rho, \rho) = 0$, the deformation theory of $\rho$ is smooth, or "unobstructed." This well-known phenomenon is visible in Theorem 3.1.1 in the case $r = 1$ (i.e. $\rho$ is irreducible), and remains the case for general $r \geq 1$ when we study the stack of representations $\text{Rep}_G$ mentioned in §3.1. We call the case $\text{Ext}^2_G(\rho, \rho) = 0$ the "representation-unobstructed case" for clarity, when our focus is on the deformation theory of pseudorepresentations.

When $r > 1$, it is possible for $R_D$ to be non-regular even when $\text{Ext}^2_G(\rho, \rho) = 0$; actually, $R_D$ is rarely regular. But other ring-theoretic properties hold, which arise from invariant theory.

**Corollary 3.3.13** (The representation-unobstructed case). Assume $\text{Ext}^2_G(\rho, \rho) = 0$, i.e. $H^2(G, \text{End}_F(\rho)) = 0$. Then the surjection $R_D^1 \twoheadrightarrow R_D$ of Theorem 3.3.1 is an isomorphism. In particular,

1. there exists an isomorphism

$$t_D \sim \bigoplus_{\gamma \in \mathcal{S}(r)} \Sigma \text{Ext}^1_G(\gamma),$$

3. When $\mathcal{C}(D)$ is strongly connected, the Krull dimension of $R_D$ is

$$\dim R_D = 1 - r + \sum_{i,j \in r} h^1_{ij}.$$  

4. When $\mathcal{C}(D)$ is not strongly connected, then $R_D \cong \bigotimes_a R_{D_a}$ and $\dim R_D = \sum_a \dim R_{D_a}$, where $D = \bigoplus_a D_a$ is the decomposition of $D$ into strongly connected summands.
5. An $n$-th order pseudodeformation $D_n$ of $D$ extends to an $(n+1)$-st order pseudodeformation if and only if the obstruction class $\alpha(D_n)$ of Corollary 3.3.11 vanishes. That is, the obstruction $\beta(D_n)$ is always zero, when it exists.

**Proof.** The first statement is clear in Theorem 3.3.1. The second statement follows the theorem combined with Fact 3.3.1. □

**Remark 3.3.14.** While $R_D^1$ is Cohen-Macaulay, for general $r$ and $h^1_{ij}$, it is very rare for $R_D^1$ to be Gorenstein. It is even more rare for it to be complete intersection or regular. See the discussion of §11.6. But for small $r$ and $h^1_{ij}$, there are some of these cases, and they are well-understood: see Example 11.6.3.

**Remark 3.3.15.** The Taylor–Wiles method [TW95], and subsequent developments, involve auxiliary deformation problems where one arranges for $H^2(\text{aux}, \text{End}_F(\rho))$ to vanish. This often goes under the moniker "killing the dual Selmer group." This has the effect of making deformation rings $R^\text{aux}_D$ isomorphic to a power series ring. We see in Corollary 3.3.13 what can be deduced about pseudodeformation rings $R_D$ from killing the dual Selmer group. This should be compared with the philosophy that, in situations where $\ell_0 = 0$ so that the Taylor–Wiles method could possibly be applied (see e.g. [CG18] for more on this, including the definition of $\ell_0$), local Galois deformation rings – when properly set up to correspond to Hecke algebras – ought to be at least Cohen-Macaulay in situations where $\ell_0 = 0$. This is proved in e.g. [Sno18 Thm. 4.6.2].

We conclude with bounds on the Krull dimension of $R_D$. 
**Corollary 3.3.16** (Bounds on Krull dimension). Assume for simplicity that $C(D)$ is strongly connected. Then we have the following bounds on the Krull dimension of $R_D$. Letting $h_D^1 := 1 - r + \sum_{i,j \in r} h_{ij}^1 = \dim R_D^1$, we have

$$h_D^1 - \sum_{\gamma \in SC(r)} h_{ij}^2 \cdot h_1^1(\gamma) \leq \dim R_D \leq h_D^1$$

**Remark 3.3.17.** Note that $h_1^1(\gamma')$ is multiplicative in the $h_{ij}^1$, while $h_D^1$ is additive in the $h_{ij}^1$. Therefore, for large dimensions of Ext$^1$-groups in the presence of non-zero Ext$^2$-groups, the lower bound on $\dim R_D$ is trivial.

**Remark 3.3.18.** We remark on the computations involved in the proof of Theorem 3.3.1. The ring $R_D^1$ is the invariant subring of the codomain of (3.2.5) under the adjoint co-action of the $F$-algebraic torus induced by units of $\End_F[G](\rho) \sim \bigoplus_{i=1}^r \End_F[G](\rho_i) \cong F^r$.

Similarly, as this action is linearly reductive (in any characteristic), the presentation of $R_D$ in Theorem 3.3.1 follows via a calculation of invariants from Theorem 11.3.1, which generalizes Theorem 3.1.1 to the case that $\rho$ is semi-simple with distinct simple factors.

### 3.4. Amplification: Galois representations with conditions.

In this section, we state a meta-result: all of the theorems and corollaries of §3.1 and §3.3 may be applied to deformation rings and pseudodeformation rings parameterizing Galois representations satisfying additional Galois-theoretic conditions of certain kinds.

Let $C$ be a condition that applies to finite-length $F[G]$-modules. We say that $C$ is a stable condition when the full subcategory of finite-length $F[G]$-modules satisfying $C$ is closed under the formation of subquotients and finite direct sums. It has been understood, since the work of Ramakrishna [Ram93], that there exists a quotient $R_\rho \twoheadrightarrow R_C^\rho$ parameterizing exactly those deformations with property $C$.

For stable $C$, and some other specified classes of conditions $C$, the author’s joint work with Wake [WWE17a] (see §12.1 for a summary) explains that

1. there exist quotient algebras of $F[G]$ factoring the action on representations with residual pseudorepresentation $D$ and condition $C$, and
2. there exists a sensible notion of “pseudorepresentation of $G$ with property $C$” and a quotient $R_D \twoheadrightarrow R_C^D$ parameterizing exactly those pseudodeformations with property $C$.

**Theorem 3.4.1.** Let $C$ be a stable condition on finite length $F[G]$-modules. Let $\rho$ be a $F$-linear representation of $G$ that is semi-simple with distinct absolutely irreducible factors. Then there exists a dg-subalgebra

$$C^\bullet(C(\rho), \End_F(\rho)) \subseteq C^\bullet(G, \End_F(\rho)),$$

and the $A_\infty$-algebra structure of Fact 2.2.4 on the cohomology $H^\bullet(C(\rho), \End_F(\rho)) \cong \Ext^\bullet_C(\rho, \rho)$ of $C^\bullet(C, \End_F(\rho))$ induces

1. when $\rho$ is irreducible, a presentation of $R_C^\rho$, as in Theorem 3.1.1, and
2. when $D$ is the pseudorepresentation induced by $\psi$, a presentation of $R_C^D$, as in Theorem 3.3.1.
All of the corollaries to these theorems also apply to these theorems, as they hinge only on the existence of the \(A_\infty\)-algebra structure on cohomology and its relation to the deformation rings.

**Example 3.4.2.** Stable conditions \(\mathcal{C}\) of interest in the study of Galois representations include such conditions as

1. when \(G = G_F\) is the absolute Galois group of a \(p\)-adic field \(F\), or when \(G\) admits a homomorphism \(G_F \to G\), we ask for the property that the \(G_F\)-action arises from the \(\mathcal{T}\)-points of a finite flat group scheme defined over the ring of integers \(O_F\) of \(F\).

2. More generally than (1) when \(F/\mathbb{Q}_p\) is unramified, there are Fontaine–Laffaille conditions, which are \(p\)-integral crystalline conditions.

**Remark 3.4.3.** For a choice of subcategory \(\mathcal{C}\) of the category of finite-length \(\mathbb{F}[G]\)-modules, there may exist a notion of \(\operatorname{Ext}_\mathcal{C}^\bullet(\rho, \rho')\) for \(\rho, \rho' \in \mathcal{C}\). We emphasize that the \(\operatorname{Ext}_\mathcal{C}^\bullet(\rho, \rho')\) above may not be the same as \(\operatorname{Ext}_\mathcal{C}^\bullet(\rho, \rho')\) in all degrees. It is, however, the same in degrees 0 and 1. In degree 2, we have

\[
\operatorname{Ext}_\mathcal{C}^2(\rho, \rho) \subset \operatorname{Ext}_\mathcal{C}^2(\rho, \rho).
\]

Nonetheless, the \(A_\infty\)-algebra labeled by \(\mathcal{C}(\rho)\) correctly calculates \(R_C\) and \(R_{CD}\). In fact, this is a general observation (that isomorphism in degrees 0 and 1 and injection in degree 2) result in the same classical deformation problem, see e.g. [GM88, Thm. 2.4]. We explain how this difference arises in Remark 12.3.2.

**Remark 3.4.4.** Let \(G = G_K,S\), the Galois group of a number field \(K\) ramified at a finite set of places \(S\). Let \(G_v \to G\) represent decomposition groups at finite places \(v \in S\). Assume that the subcategory \(\mathcal{C}\) can be expressed as a set of conditions \(\mathcal{C} = \{\mathcal{C}_v \mid v \in S\}\), where \(\mathcal{C}_v\) is applied to the restriction of \(\hat{\rho}|_{G_v}\) of a deformation \(\hat{\rho}\) of \(\rho\). Extending Remark 3.4.3 one might naturally ask whether \(\operatorname{Ext}_\mathcal{C}^1(\rho, \rho)\) can be shown to be canonically isomorphic to Selmer groups, that is,

\[
\operatorname{Sel}(\mathcal{C}(\rho), \mathcal{L}_v) := \ker \left( H^1(G, \text{End}_{\mathbb{F}}(\rho)) \to \bigoplus_{v \in S} \frac{H^1(G_v, \text{End}_{\mathbb{F}}(\rho))}{\mathcal{L}_v} \right),
\]

for subspaces \(\mathcal{L}_v \subset H^1(G_v, \text{End}_{\mathbb{F}}(\rho))\) corresponding to the condition \(\mathcal{C}_v\).

We defer the question in this generality to future work. However, see [13] for examples of Selmer conditions that can be proved to be realized by \(\operatorname{Ext}_\mathcal{C}^1(\rho, \rho)\). See also [12] for the existence of dg-algebra structures on suspended cones of morphisms of dg-algebras and the resulting relative representability of deformation problems parameterized by this cone. This is relevant to this Selmer case, as Nekovář has shown that Selmer groups arise as the cohomology of cones in some cases [Nek06].

### 4. Complements

In this section we discuss an alternate formulation of the main theorems in terms of Massey products, examples that illustrate the main theorems and computations of the paper, relationships with other works in number theory, related and/or antecedent works outside number theory, and acknowledgements of the influence of colleagues. Here is a list of contents.

- Massey products and their relationship to \(A_\infty\)-products, and previous appearances of Massey products in number theory in work of Sharifi [Sha07].
The computations of ranks of p-adic modular Hecke algebras in terms of $A_\infty$-products that appear in §15.

The norm residue isomorphism theorem.

The derived Galois deformation rings of Galatius–Venkatesh [GV18], and the analogue of the cotangent complex in our setting.

Non-commutative geometry and deformation theory.

The Kuranishi map.

Acknowledgements.

In particular, we point out in §4.5 how the content of Part 2 of this paper relies on and also advances the line of inquiry in non-commutative deformation theory pursued by Laudal [Lau02] and Segal [Seg08].

4.1. Massey products. Massey products and their defining systems provide an alternative to $A_\infty$-structures for the purposes of this paper. (See §8 for an introduction to Massey products.) This is true in a formal sense: the main theorems stated above are the outcome of Part 3 of this paper, which in turn relies on results in non-commutative algebraic geometry in Part 2. The main result of Part 2, Corollary 7.4.5 gives a presentation of the completion $\mathbb{F}[G]_\rho^\wedge$ of $\mathbb{F}[G]$ at the kernel of $\rho$ in terms of $A_\infty$-structures and certain choices of idempotents. In comparison, the main theorem of [Lau02] gives an expression of the same algebra in terms of Massey products. We make further comparisons with previous work of Laudal and Segal in §4.5.

The definition of a Massey product, their defining systems, and the relationship between these and $A_\infty$-algebras is given in §8. This is applied to non-commutative deformation theory in §9. The purpose of §9 is different than the rest of the paper: it illustrates how Massey products naturally arise when doing explicit computations of deformations, and it also illustrates how using $A_\infty$-algebras are more convenient for presenting deformation rings.

We discuss other studies of Massey products in Galois cohomology in §4.3. Also, Massey products in the Galois cohomology of an endomorphism algebra has been connected to deformations of Galois representations and ranks of Hecke algebras in [WWE17c]. We discuss this next.

4.2. Ranks of Hecke algebras. In §13 as an example of an application of our main results, we determine the ranks of some p-adic modular Hecke algebras in terms of the presentations given by $A_\infty$-products. This relies on a known isomorphism $R^C_{\rho} \cong \mathbb{T}/m_\mathbb{T}$, where

- $\mathbb{T}$ is the Hecke algebra in question,
- $m$ is the maximal ideal, with residue field $\mathbb{F}$, of the regular local base ring over which $\mathbb{T}$ is known to be free and for which we are measuring the rank of $\mathbb{T}$, so we can calculate $\text{rank } \mathbb{T} = \dim_\mathbb{F} R^C_{\rho}$
- “*” in $R^C_{\rho}$ stands in for either
  - $\rho$, a 2-dimensional absolutely irreducible $\mathbb{F}$-valued representation of a global Galois group $G$, which the residual semi-simplification associated to the Hecke eigensystem modulo $\rho$ cut out by the maximal ideal of $\mathbb{T}$; or
  - $D$, a 2-dimensional pseudorepresentation given by $D = \psi(\rho)$ for some 2-dimensional representation $\rho$ of $G$ such that $\rho \cong \chi_1 \oplus \chi_2$ where $\chi_1 \neq \chi_2$, determined similarly by the residual Hecke eigensystem of $\mathbb{T}$
- $\mathcal{C}$ is a stable condition on finite-length $\mathbb{Z}_p[G]$-modules
\( R^C_C \) is a deformation (resp. pseudodeformation) ring of \( \rho \) with condition \( C \).

This rank is an expression of size of a congruence class of modular eigenforms modulo \( p \). The result of each example given in §13 is an expression of rank \( T \) in terms of an arithmetic invariant expressed in terms of the vanishing of \( A_\infty \)-products (or, equivalently, Massey products).

The first two examples that we give in §13 are drawn from the finite-flat case and the ordinary case of Wiles’ \( R \cong \mathbb{T} \) theorem [Wil95]. In contrast, the third example is residually reducible, having to do with the Galois representations and modular forms appearing in Ribet’s proof of the converse to Herbrand’s theorem [Rib76].

It is especially simple to calculate this rank when the tangent space of \( R^C_C \) is known to have dimension 1. Then we know that \( R^C_C \cong \mathbb{F}[\epsilon]/(\epsilon^{n+1}) \) so that rank \( T = n+1 \), where \( n \) is the greatest \( i \geq 1 \) such that the first-order deformation of \( \star \) given by a non-zero tangent vector \( \tau \) extends to an \( i \)-th order deformation. In the residually irreducible case \( \star = \rho \), this is simply the largest \( n \geq 2 \) such that the \( n \)-th \( A_\infty \)-power \( m_n(\tau^{\otimes n}) \) of \( \tau \in \text{Ext}^2_C(\rho, \rho) \) in \( \text{Ext}^2_C(\rho, \rho) \) is equal to zero. Or, equivalently, it is the largest \( n \geq 2 \) for which the \( n \)-th Massey power \( \langle \tau \rangle^n \subset \text{Ext}^2_C(\rho, \rho) \) is defined and contains zero. In particular, the case \( n = 2 \) is unambiguously defined, as noted in Remark 3.1.4: a cup product determines whether \( n = 2 \) or \( n > 2 \).

In the residually reducible case where the tangent dimension is 1, the \( A_\infty \)-products or Massey products that must be calculated to determine \( n \geq 1 \) such that \( R^C_D \cong \mathbb{F}[\epsilon]/(\epsilon^{n+1}) \) order of vanishing depend on whether the tangent vector is reducible or not, in the sense of Definition 11.5.1. We give an example of the irreducible case in §13.3 which is related to [Rib76] as mentioned above. In this particular case, the tangent dimension is at most 1 upon Vandiver’s conjecture, and the tangent dimension is zero if and only if \( p \) does not divide an appropriate Bernoulli number.

These examples are very similar to but do not include the case of the residually Eisenstein Hecke algebra of Mazur’s paper on the Eisenstein ideal [Maz77], whose rank is determined in terms of Massey products in [WWE17c]. The reason for this is that the relevant deformation condition cannot be expressed as a stable condition \( C \). The author expects that there is an appropriate generalization of Theorem 3.4.1 that can encompass these more general conditions. For the moment, this gives an example of the flexibility of Massey products. We discuss it in §13.4 also answering there a question stated in [WWE17c], constructing a Massey product in “finite-flat cohomology.”

4.3. Galois cohomology of the trivial representation. The universal commutative deformation ring of a character (i.e. 1-dimensional representation) is relatively straightforward: it is simply a completed and abelianized group algebra, as pointed out by Mazur [Maz89, §1.4]. The non-commutative deformation rings we discuss in Part 2 are completed but not abelianized. In connection with this, the non-commutative version of the cohomological expression of this deformation ring of Theorem 3.1.1 is still very much of interest: it is, after all, equal to the modulo \( p \) pro-unipotent completion of the profinite group \( G \) and related to the \( p \)-adic pro-unipotent completion (which could also be studied using the tools of this paper).

As one can see from the explicit way of writing down deformations in §9.2, deformations of the trivial character are simply Heisenberg group-valued representations with extra symmetry.
If we let $G = G_F$ be the absolute Galois group of a field $F$ with characteristic different than $p$ and containing the $p$-roots of unity, then, upon a choice of $p$th root of unity, the dg-algebra

$$\bigoplus_{i \geq 0} C^i(G_F, \mu_p^{\otimes i}) \cong C^i(G_F, \mathbb{F}_p)$$

has cohomology $\mathbb{F}_p$-algebra whose form is given by Milnor $K$-theory according to the norm residue isomorphism theorem of Rost and Voevodsky [Voe11] (i.e. the proved motivic Bloch–Kato conjecture, or Milnor conjecture when $p = 2$). As $\mathbb{F}_p \cong \text{End}_{\mathbb{F}_p}(\rho)$ for any 1-dimensional representation $\rho$ over $\mathbb{F}_p$, the $A_\infty$-structure (or higher Massey products) enrich this ring and retain extra information. Since the work of Hopkins–Wickelgren [HW15], there has been attention to the vanishing of higher Massey products on $H^1(G_F, \mathbb{F}_p)$ and their links with the arithmetic of $F$. For example, as proved in [HW15, MT17] triple Massey products on $H^1(G_F, \mathbb{F}_p)$ vanish. There has also long been interest in determining the structure of pro-$p$ completions of Galois groups, which is clearly very much related. The interested reader can look into the extensive literature on these topics; the introduction of [MT17] contains a survey.

We observe that the natural $A_\infty$-structure on cohomology $H^*(G_F, \text{End}_F(\rho))$ for arbitrary $\rho$ provides the setting for an “unstable” generalization of these questions, at least when $F$ contains the $p$-th roots of unity. By “unstable,” we mean that $\rho$ becomes trivial after restriction to a finite subgroup of $G_F$. Indeed, just as the existence of $F$-points on an algebraic variety is connected with the vanishing of a triple Massey product in [HW15], the study of deformations of Galois representations has been motivated by its applications to arithmetic algebraic geometry.

In contrast to the setting of the norm residue isomorphism theorem, the study of Galois groups of global fields with restricted ramification $G_{F,S}$ is quite different. For one thing, the norm residue isomorphism theorem does not apply. It also is understood, for example, that there are non-vanishing triple Massey products. See in particular [HW15, Ex. 2.11], which is due to Gärtner, and the references in [HW15, §1].

Similarly, Sharifi [Sha07] works over the cyclotomic $\mathbb{Z}_p$-extension of a number field, so that $p$-power Kummer extensions appear in the first cohomology of the trivial representation. Thus the work of Sharifi can also be interpreted deformation-theoretically as deformations of the trivial character. He relates the vanishing behavior of certain Massey products (in cohomology with restricted ramification) to Iwasawa-theoretic class groups.

### 4.4. The derived deformation rings of Galatius–Venkatesh.

We draw some comparisons between the approaches to deformation theory in present paper and the work of Galatius–Venkatesh [GV18] on derived Galois deformation rings. On the way to doing this, we explain the limitations and advantages of the setting of $A_\infty$-algebras chosen in this paper.

The dg-algebra $C^*(G, \text{End}_F(\rho))$ and the $A_\infty$-algebra structure on its cohomology have the information of a derived enrichment of conventional deformation theory that we study in this paper. The coefficient rings of this enrichment are $\mathbb{F}$-augmented dg-Artin algebras that are associative but not commutative in any sense.
To this author’s knowledge, the most straightforward example of an exposition of such a derived enrichment is work of Kapranov [Kap01]; this applies to local systems on a finite CW-complex valued in an affine algebraic group over $\mathbb{C}$, with coefficient rings in *commutative dg-$\mathbb{C}$-algebras*.

The contrast in settings between the present work and [Kap01] are indicative of the reasons for our choice of setting.

- In positive characteristic or mixed characteristic, where the desired applications of this paper are located, formulations of Koszul duality between Lie and commutative operads are topics of contemporary homotopy-theoretic research. In contrast, the associative operad is well-understood to be self-dual in any characteristic. In this introduction, Koszul duality of operads is visible in the bar construction of $[\mathcal{A}_\infty]$ it sends an $A_\infty$-algebra to a dg-coalgebra, but an analogue sends an $L_\infty$-algebra (the Lie version of $A_\infty$) to a commutative dg-coalgebra (see e.g. [LV12]). However, commutative dg-algebras are only well-behaved in characteristic zero. Therefore, we have written $\text{End}_F(\rho)$ instead of $\text{ad}\rho$ to emphasize that we choose the associative algebra structure on the endomorphism ring, as opposed to its induced Lie algebra structure. Accordingly, our strategy to determine commutative deformation rings and other moduli spaces with commutative coefficients is to stay in the associative (non-commutative) world at least until Koszul duality is applied, and then finally abelianize at the end.

- Consequently, we do not work with representations valued in general algebraic groups, instead focusing on matrix-valued representations. For the cohomology controlling the deformations of algebraic group-valued representations is intrinsically valued in a Lie algebra that is not naturally induced by an associative algebra.

- We also work in constant characteristic. Some additional Bockstein type map is needed to control deformations to mixed characteristic.

In contrast, Galatius–Venkatesh [GV18] work with algebraic groups $G$ other than $GL_n$ and mixed characteristic coefficient rings. Therefore, commutative dg-algebras do not suffice as coefficient rings. They work with coefficients in simplicial commutative rings and study $G$-valued local systems on certain étale homotopy types. What we gain from a more limited choice of setting than [GV18] is the concreteness of the calculations and the presentations for objects of interest: ultimately, we are just doing computations with functions on the profinite group $G$.

Indeed, one of the advantages of working with a well-formulated derived deformation problem is that the cotangent complex of the deformation problem for $\rho$ is realized by $C^\bullet(G, \text{End}_F(\rho))$ and the André–Quillen cohomology of the derived deformation ring is realized by $H^\bullet(G, \text{End}_F(\rho))$ [GV18, Lem. 5.10]. In the present work, we take these as our starting point.

4.5. **Non-commutative deformation theory.** As we have mentioned above, Part 2 has a new result in non-commutative deformation theory that is applied in Part 3, along with some results of [WE18], to prove the main theorems stated in this introduction. The main results of Part 2 are Theorem 7.4.3 and Corollary 7.4.5. We want to make some comments about how these results are related to other work in non-commutative deformation theory.
In non-commutative deformation theory, the content of these results addresses what is called the “deformation theory of \( r \)-points.” Here we have an associative \( \mathbb{F} \)-algebra \( E \). A “point” of \( E \) is a maximal ideal \( m \) of \( E \), and we extend \( \mathbb{F} \) if necessary so that \( E/m \cong M_d(\mathbb{F}) \). We now take \( r \) points, thought of as surjective representations \( \rho_i : E \to M_d(\mathbb{F}) \) for \( i = 1, \ldots, r \) cutting out distinct maximal ideals. One principal distinction from commutative deformation theory is that distinct points can have extensions between them, i.e. \( \text{Ext}^1_E(\rho_i, \rho_j) \) can be non-trivial when \( \rho_i \not\cong \rho_i \). This does not happen in the commutative setting. Let \( \rho := \bigoplus_{i=1}^r \rho_i \) as usual. In doing associative deformation theory, we are interested in determining the completion \( E_{\rho}^\wedge \) of \( E \) at the kernel of \( \rho \).

The main results follow on previous work of Segal \cite{Segal08}, which in turn is a development of work of Laudal \cite{Laudal02}. Segal proves a result that is very similar to Corollary 7.4.5(2); in \cite[Thm. 2.14]{Segal08}, he proves that what we call \( R_{nc}^\rho \) is isomorphic to what we call \( R \) (in the notation of Corollary 7.4.5). The difference is that we keep track of data that determines an isomorphism between \( R_{nc}^\rho \) and \( R \), and determines a presentation for \( E_{\rho}^\wedge \) in terms of cohomology. This is the data of a homotopy retract between the Hochschild cochain complex \( C^*(E, \text{End}_\mathbb{F}(\rho)) \) and its cohomology, which we explain in \S 5.2. We also carefully keep track of the some choices of idempotents that we use to remove Segal’s assumption that \( d_i = 1 \) for all \( i \). Thus we have identified data that determines a presentation of \( E_{\rho}^\wedge \) and the deformation functor, while Segal’s approach identifies the isomorphism class of \( E_{\rho}^\wedge \) and its deformation functor. Laudal \cite{Laudal02} also characterizes the isomorphism class of \( E_{\rho}^\wedge \) in terms of Massey products. Thus \( E_{\rho}^\wedge \) is described by Laudal in an inductive way, as we discuss more in \S 9.3.

As we carry this out, we are careful in \S 5.6 to make sure that notions of non-commutative gauge equivalence of Maurer-Cartan elements correspond to conjugacy classes of representations. This is well-understood in characteristic zero (see e.g. \cite[Defns. 2.3 and 2.9]{Segal08}), but appears less often in the literature in arbitrary characteristic because it is often expressed as an exponential. For this purpose, we found Prouté’s study \cite{Pro11} of twisting morphisms very useful, as well as \cite{CL17}. The statement of the decomposition theorem (Theorem 5.3.3) for \( A_{\infty} \)-algebras by Chuang–Lazarev \cite{CL17} was also very helpful.

Finally, we remark that the text by Le Bruyn \cite{LB08} summarizes many results from non-commutative geometry related to the content of Part 2 and also the Cayley-Hamilton algebra theory and pseudodeformation theory dealt with in Part 3, especially focusing on the representation-unobstructed case (in the terminology of Corollary 3.3.13). This is especially relevant (see \S 3.2 and \S 11.6) for the study of the ring \( R_P^\wedge \) that appears in the main Theorem 3.3.1.

4.6. The Kuranishi map. We discuss one more perspective on the presentation of the deformation ring in terms of cohomology in Theorem 3.1.1. These have been studied in the context of the variation or deformation of flat connections on manifolds. In this case, there are functions whose analytic germ is the denominator in the expression of the deformation ring of Theorem 3.1.1 Indeed, this germ has a natural extension to an analytic function in the neighborhood of the origin in the appropriate cohomology vector space \( H^1(\text{End}(\rho)) \), known as the Kuranishi obstruction map; see e.g. \cite[Ch. 12]{MMR94}. See also \cite{GM88} for the connection between such moduli spaces of connections and moduli spaces of representations. In \cite{GM88}, when \( G \) is the fundamental group of a compact Kähler manifold, the
comparison between dg-Lie algebras $C^\bullet(G, \text{End}_C(\rho))$ and the dg-Lie algebra controlling deformations of the connection is exploited by Goldman–Millson to prove that they are formal. This has a consequence that the Lie algebra $H^\bullet(G, \text{End}_C(\rho))$ loses no information, i.e. the higher $L_\infty$-bracket structures $\ell_n$ for $n \geq 3$ may be chosen to vanish, and the presentation for the deformation space as in Theorem 3.1.1 is quadratic.

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4.8. Notation and terminology. $G$ denotes a profinite group. $F$ denotes a finite field of characteristic $p$, in which $F[G]$ has the standard profinite topology with completion $\hat{F}[G]$. The topology of the codomain of functions with domain $G$ or $F[G]$ is always from a presentation as a finitely generated (left) module over a topological $F$-algebra. These topological $F$-algebras are

- $\mathcal{A}_F$, the category of Artinian local associative $F$-algebras with residue field $F$, equipped with the discrete topology;
- $\mathcal{C}_F$, the full subcategory of $\mathcal{A}_F$ consisting of commutative objects;
- the categories of limits $\hat{\mathcal{A}}_F$ and $\hat{\mathcal{C}}_F$, with the resulting profinite topology;
- $\mathcal{Aff}_F$, the category of topologically finitely generated $F$-algebras; this is the opposite category to the category of Noetherian affine Spf $F$-formal schemes (see [Gro60, §10.1]).

$E$ denotes an associative $F$-algebra. Often we consider the case $E = F[G]$ or variants of this, in which case $E$ is topological as discussed above.

$(V, \rho)$ denotes a finite-dimensional representation of $E$, i.e. a finite-dimensional $F$-vector space $V$ with a left $F$-linear action $\rho$ of $E$. We write $\text{End}_F(\rho)$ for the adjoint representation of $\rho$, which is an $E$-bimodule. (We write “$R$-bimodule” as shorthand for “$(R,R)$-bimodule.”) In contrast, we use $\text{End}_F(V)$ for the same $F$-vector space, but in this case emphasizing its $F$-algebra structure which receives the homomorphism $\rho$. The difference is only a matter of emphasis. We also write “$\rho$” when $V = F^d$, writing $\rho$ as a homomorphism

$$\rho : E \rightarrow M_d(F) \quad \text{or} \quad \rho : G \rightarrow \text{GL}_d(F)$$

in this case.
Write $F[\epsilon_n]$ for $F[\epsilon]/(\epsilon^{n+1}) \in C_F$. Given a homomorphism $\rho : G \to \text{GL}_d(F)$, we use the term “$n$-th order lift” to describe a homomorphism $G \to \text{GL}_d(F[\epsilon]/\epsilon^{n+1})$ that reduces to $\rho$ modulo $\epsilon$. In contrast, an “$n$-th order deformation” refers to an orbit of lifts under the adjoint action of $\text{ker}(\text{GL}_d(F[\epsilon]/\epsilon^{n+1})) \to \text{GL}_d(F)$. We use the same terms for lifts of representations of $E$.

The term “$F^r$-algebras” refers to algebras algebras in the category of $F^r$-bimodules. In particular, this does not refer to algebras receiving a map from $F^r$ to the center; see §7.1 for more on this. We write $\otimes$ for the tensor product in the category of $F^r$-bimodules, which is also the tensor product for $F^r$-algebras. All of this reduces to the usual setting of $F$-algebras when $r = 1$. So we will refer to $F^r$-algebras for the rest of this introduction to notation.

Graded objects are indexed by $\mathbb{Z}$. Derivations and differentials on a graded object have degree 1 unless otherwise stated. A complex is a graded object with a differential: we use the notation $(C,d)$ for a complex, where $d : C \to C^{i+1}$. Suspensions $\Sigma$ on a complex produces the complex where $\Sigma C^i = C^{i+1}$ and $d_{\Sigma C} = -d_C$.

We write “dg-algebra” for a differential graded algebra. These are complexes equipped with an associative multiplication satisfying the Leibniz rule, and are denoted $(C,d_C,m_{2,C})$. These are most often dg-$F^r$-algebras.

We refer to augmented dg-$F^r$-algebras $C$, meaning that there is a augmentation map $C \to F^r$. A complete dg-$F^r$-algebra is an augmented dg-$F^r$-algebra that is complete with respect to the kernel of the augmentation map. A free complete algebra (resp. graded algebra, resp. dg-algebra) on a (resp. graded, resp. dg) $F^r$-module $V$ is the (resp. graded, resp. dg) tensor algebra

$$\hat{T}_{F^r}V := \prod_{n \geq 0} V^\otimes n$$

(resp. equipped with the differential produced by extension via the Leibniz rule).

We use the following notation for categories of dg-algebras.

- $\mathcal{A}_{F^r}$ finite-dimensional augmented dg-$F^r$-algebras
- $\mathcal{A}_{F^r}$ limits of finite-dimensional augmented dg-$F^r$-algebras, such as $\hat{T}_{F^r}$ when $V$ has finite dimension as an $F$-vector space.

Undecorated tensor products “$\otimes$” are assumed to be over $F$. Likewise, we use $(-)^\ast$ to denote the $F$-linear dual of a $F$-linear object or morphism. This applies naturally to objects with $F$-bimodule structure as well. On a graded $F$-vector space $C$, it is applied graded-piecewise by default: $(C^\ast)^i := (C^{-i})^\ast$. We refer to this as a “graded-dual” object when we wish to emphasize this. The duality operation on a complexes produces a complex $(C^\ast,d_{C^\ast})$ where $d_{C^\ast} := (d_C^{i-1})^\ast$. As is standard when working with tensor products of morphisms of graded vector spaces, the Koszul sign rule

$$(f \otimes f')(x \otimes x') = (-1)^{\vert f \vert \vert x \vert} f(x) \otimes f'(x')$$

is in force.

Coalgebras inherit all of the notions above: codifferential, coaugmentation, co-complete, cofree, etc., in the standard way.

Given a graded $F$-vector space $C$, we will often refer to the graded vector space denoted $\Sigma C^\ast$. This is shorthand for “$(\Sigma C)^\ast$”: it is the graded-dual of the suspension.
Part 2. $A\infty$-algebras and deformation theory

In Part 2, we develop the theory of $A\infty$-algebras, use them to produce presentations of associative deformation rings, and discuss their relationship with Massey products.

5. $A\infty$-algebras

In this section we recall the definition of an $A\infty$-algebra and discuss the relationship between $A\infty$-algebras and dg-algebras. We also discuss the bar equivalence between $A\infty$-algebras and cocomplete cofree dg-coalgebras, which will provide a useful perspective on $A\infty$-algebras in the sequel.

5.1. Defining $A\infty$-algebras. Recall that undecorated tensor products $\otimes$ are over the base field $F$.

Definition 5.1.1. An $A\infty$-algebra over $F$ is a pair $(A, (m_n)_{n \geq 1})$ consisting of a graded $F$-vector space $A = \bigoplus_{i \in \mathbb{Z}} A^i$ and a sequence of homogenous degree 2 maps $m_n : A^\otimes n \rightarrow A$, $n \geq 1$ such that

\[
\sum (-1)^{r+s+t} m_u(1^\otimes r \otimes m_s \otimes 1^\otimes t) = 0,
\]

where the sum ranges over all decompositions $n = r + s + t$ into non-negative integers, with the conventions that $u = r + 1 + t$ and $m_0 = 0$. We also use $m$ as shorthand for $(m_n)_{n \geq 1}$.

A morphism $f : A \rightarrow A'$ of $A\infty$-algebras $(A, m)$, $(A', m')$ is a sequence of maps $f_n : A^\otimes n \rightarrow A'$, $n \geq 1$ of homogenous degree $1 - n$ such that, for all $n \geq 1$,

\[
\sum (-1)^{r+s+t} f_u(1^\otimes r \otimes m_s \otimes 1^\otimes t) = \sum (-1)^s m'_u(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_n})
\]

where the first sum runs over all decompositions $n = r + s + t$ into non-negative integers, we maintain $u = r + 1 + t$, and the second sum runs over all $1 \leq r \leq n$ and all decompositions $n = i_1 + \cdots + i_r$ into positive integers, where $s = \sum_{j=1}^{r-1} j(i_j - 1)$.

In particular, the relations above imply that $m_1^2 = 0$, i.e. $(A, m_1)$ is a complex. They also imply that $f_1$ is a morphism of complexes $(A, m_1) \rightarrow (A', m'_1)$. A morphism $f$ of $A\infty$-algebras is called a quasi-isomorphism when $f_1$ is a quasi-isomorphism of complexes. It is called an isomorphism when $f_1$ is an isomorphism of complexes; indeed, $f$ is an isomorphism if and only if it has an inverse.

An $A\infty$-structure records homotopies that make $m_2$ “associative up to homotopy,” in contrast to a dg-algebra, which are (strictly) associative.

Example 5.1.4. The following examples illustrate the relationship between this definition and differential graded algebras. Let $(A, m)$ be an $A\infty$-algebra.

1. We have noted that $[5.1.2]$ states that $(A, m_1)$ is a complex.
2. For $n = 2$, $[5.1.2]$ states that $m_2(m_1 \otimes 1 + 1 \otimes m) = m_1 m_2$, the Leibniz rule.
Example 5.2.2. Let \((C,d)\) be a cochain complex. Then \((H^{\bullet}(C),0)\) is a homotopy retract of \((C,d)\), when it is equipped with maps \(i,p\) set up in the following way.
For each \( n \), choose a section \( i^n : H^n(C) \to C^n \) of the standard map from \( \ker(d_{C|C^n}) \) to \( H^n(C) \) and a section \( h^n : d_{C|C^n} \to C^n \) of \( d_{C|C^n} : C^n \to C^{n+1} \). These sections produce a decomposition \( C^n \cong \im(d_{C|C^n}) \oplus i^n(H^n(C)) \oplus h^n(d_{C|C^n}(C^n)) \), which we write as

\[
C^n = B^n(C) \oplus \tilde{H}^n(C) \oplus L^n(C) = B^n \oplus \tilde{H}^n \oplus L^n.
\]

Think of the three summands as

- coBoundaries,
- cocycles lifting \( H^n(C) \), and
- Lifts of coboundaries to a cochain inducing it.

Then let \( p^n : C^n \to H^n(C) \) be the projection killing the summands complementing \( \tilde{H}^n \). Combine the above data into maps of complexes \( p \) and \( i \). Finally, extend \( h^n \) to \( C^{n+1} \) by killing the summands complementing \( B^{n+1} \), so that the \( h^n \) together produce a map \( h : C \to C[1] \) of graded vector spaces. By design, \( p \), \( i \), and \( h \) make \((H^*(C), 0)\) a homotopy retract of \((C, d_C)\).

There is also a complimentary complex to \( H^*(C) \), which we call \( K = K^*(C) \). It consists of

\[
K^n = B^n \oplus L^n
\]

with the restricted differential \( d_K = d_{C|K} \). It is acyclic and contractible, decomposing into \( L^n \cong B^{n+1} \) plus the zero map on \( B^n \).

When \((A, d_A)\) is a homotopy retract of \((C, d_C)\), there is an isomorphism \( H^*(A) \cong H^*(C) \) induced by \( i \). Consequently, when \((C, d_C)\) admits the extra structure of a dg-algebra \((C, d_C, m_C)\), there is an induced graded algebra structure on \( H^*(A) \).

Moreover, there is a natural lift of this graded algebra structure to a graded linear map \( A \otimes A \to A \), namely,

\[
m_2(x \otimes y) := pm_C(i(x) \otimes i(y)), \quad \text{for } x, y \in A.
\]

It satisfies the Leibniz rule with respect to \( d_A \) but is not associative. It is, however, associative up to a homotopy that can be expressed in terms of the homotopy retract structure. This homotopy is a map \( m_3 : A^\otimes 3 \to A \) of homogenous degree \(-1\) that satisfies the relation \( [5.1.2] \) for \( n = 3 \), see [LV12, Lem. 9.4.2]. Iterating this procedure yields the following result.

**Theorem 5.2.5 (Kontsevich–Soibelman [KS00]).** Let \((C, (m_n'))\) be an \( A_\infty \)-algebra such that the complex \((C, m_1')\) is a homotopy retract of \((A, d_A)\) via

\[
\begin{array}{c}
\xymatrix{
(C, m_1') \ar[rr]^i & & (A, d_A) \ar[ll]_p \\
(C, m_1') \ar[rr]^i & & (A, d_A) \ar[ll]_p
}
\end{array}
\]

The retract determines an \( A_\infty \)-algebra structure \((m_n)\) on \((A, d_A)\) with \( m_1 = d_A \) and a quasi-isomorphism of \( A_\infty \)-algebras \( f = (f_n) : (A, (m_n)) \cong (C, (m_n')) \) such that \( f_1 = i \).

The equality \( f_1 = i \) will be expressed as "\( f \) is a quasi-isomorphism of \( A_\infty \)-algebras extending the quasi-isomorphism of complexes \( i \)."

**Proof.** See [LV12, Thm. 9.4.14] and the references therein. \( \square \)

We will especially apply the theorem in the following case. In this statement, recall that any dg-algebra \((C, d_C, m_C)\) is an \( A_\infty \)-algebra in the sense of Example 5.1.4(4).
Corollary 5.2.6 (Kadeishvili [Kad82]). Let \((C, d_C, m_C)\) be a dg-algebra with graded algebra of cohomology \((H^*(C), 0, \tilde{m}_C)\). Then any choice of homotopy retract
\[
\begin{array}{c}
h \circ \left( C, d_C \right) \\
\xrightarrow{p}
\end{array}
(H^*(C), 0)
\]
as in Example 5.2.2 determines an \(A_{\infty}\)-structure \((m_n)\) on \(H^*(C)\) and a quasi-isomorphism of \(A_{\infty}\)-algebras \(f = (f_i) : (H^*(C), (m_n)) \to (C, d_C, m_C)\) such that
- \(m_1 = 0\),
- \(m_2 = \tilde{m}_C\), and
- \(f\) extends \(i\), that is, \(f_1 = i\).

Proof. See [LV12, Cor. 9.4.8]. We sketch this proof in the following examples, because we require the explicit expression of \(f\) in the sequel. \qed

Example 5.2.7. For \(n \geq 3\), let \(PBT_n\) be the set of planar binary rooted trees with \(n\) leaves, as in [LV12, App. C]. For each \(T \in PBT_n\), label each leaf of \(T\) with \(i\), each intermediate branch of \(T\) by \(h\), each vertex by \(m_C\), and the root of \(T\) by \(p\). With the \(j\)-th leaf corresponding to the \(j\)-th tensor factor of \(H^*(C)^{\otimes n}\), each \(T \in PBT_n\) determines a homogenous degree 2 \(- n\) map \(H^*(C)^{\otimes n} \to H^*(C)\). We set \(m_n\) to be the sum over these maps, with sign \((-1)^{s+1}\), where \(s\) is the number of leaves lying above the left of the two branches arriving at the root.

The construction of \(f = (f_n)\) is similar. We already have \(f_1 = i\). For \(n \geq 2\), we modify the labeling of \(PBT_n\) used to produce \(m_n\) by labeling the root by \(h\) instead of by \(p\). As a result, each \(T \in PBT_n\) gives rise to a homogenous degree \(n - 1\) map \(H^*(C)^{\otimes n} \to C\). We assign \(f_n\) to the sum of these maps over all \(T \in PBT_n\), with the sign convention as above.

Example 5.2.8. For an explicit formula realizing Kadeishvili’s theorem but not referring to \(PBT_n\), we reproduce [LPWZ09, p. 2021], a formula credited to Merkulov [Mer99]. Crucially for these formulas, the homotopy retract data induce an identification of \(H^n(C)\) with a subspace \(\tilde{H}^n \subset C^n\) as in (5.2.3). We will now define versions of \(m_n\) on \(C\) instead of \(H^*(C)\). We denote these by \(\tilde{m}_n\). Initially, we set \(\tilde{m}_1 = d_C, \tilde{m}_2 = m_C\), as usual.

For \(n \geq 3\) we set define a homogenous degree \(2 - n\) map \(\tilde{m}_n : C^{\otimes m} \to C\) by
\[
\tilde{m}_n = \sum_{s+t=n, s,t \geq 1} (-1)^{s+1} m_C((h \circ \tilde{m}_s) \otimes (h \circ \tilde{m}_t))
\]
where we formally define \(h \circ \tilde{m}_1\) to be \(-id_C\).

Now, for \(n \geq 1\), we may define \(m_n : H^*(C)^{\otimes n} \to H^*(C)\) as \(p \circ \tilde{m}_n \circ i^{\otimes n}\), where \(i\) stands for the inclusion \(H \hookrightarrow C\). Likewise, let \(f_n := -(h \circ \tilde{m}_n) \circ i^{\otimes n}\), resulting in \(f_1 = i\).

One may check by examining diagrams in \(PBT_n\) that the formulations of \(m_n\) (resp. \(f_n\)) in Examples 5.2.7 and 5.2.8 are equal.

Definition 5.2.10. Let \((A, m)\) be an \(A_{\infty}\)-algebra. It is called minimal provided that \(m_1 = 0\). A minimal \(A_{\infty}\)-algebra equipped with a quasi-isomorphism from \(A\) is called a minimal model of \(A\).

The \(A_{\infty}\)-algebras \((A, m)\) produced in Examples 5.2.7 and 5.2.8 are minimal \(A_{\infty}\)-algebras. The morphism \(f\) produced in each example is the minimal model, as guaranteed by Corollary 5.2.6.
5.3. **Relationship with the minimal model.** In this section, we explain the extent to which the minimal model induced by homotopy retracts is unique. We also decompose an $A_\infty$-algebra into the sum of its minimal model and a linearly contractible factor. First we record this fact.

**Lemma 5.3.1.** In the setting of Theorem 5.2.5, the projection map $p$ extends to a quasi-isomorphism of $A_\infty$-algebras, that is, there exists a quasi-isomorphism $g = (g_n)_{n \geq 1} : (C, m') \rightarrow (A, m)$ such that $g_1 = p$.

**Proof.** See [CL17, Thm. 3.9(2)]. □

This is the uniqueness statement we will use.

**Proposition 5.3.2.** Let $(C, d_C, m_2)$ be a dg-algebra with graded algebra of cohomology $(H^\bullet(C), 0, \bar{m}_2)$. Let $(i, p), (i', p')$ be two choices of homotopy retract as in Corollary 5.2.6, resulting in two pairs $(f, m)$ and $(f', m')$.

Then the morphism of $A_\infty$-algebras $h := g \circ f$ (where $g$ is as in Lemma 5.3.1)

$$h = (h_n)_{n \geq 1} : (H^\bullet(C), m) \xrightarrow{\sim} (H^\bullet(C), m').$$

is an isomorphism, and $h_1$ is the identity map on the complex $H^\bullet(C)$.

**Proof.** The claim follows from Lemma 5.3.1 and Corollary 5.2.6. □

For the statement of this theorem, recall that the homotopy retract on $(C, d_C)$ given in Example 5.2.2 produced the linearly contractible subcomplex $(K, d_K) \subset (C, d_C)$ defined in (5.2.4). The trivial $A_\infty$-structure on $K$, which we also denote by $(K, d_K)$, consists of $m_1 = d_K$ along with $m_n = 0$ for $n \geq 2$.

**Theorem 5.3.3 (Decomposition theorem).** Let $(C, m'_1)$ be an $A_\infty$-algebra such that the complex $(C, m'_1)$ is a homotopy retract of its cohomology $(H = H^\bullet(C), 0)$ via

$$h \circ (C, m'_1) \xrightarrow{p} (H, 0)$$

in the form of Example 5.2.2. Let $(H, m)$ be the minimal $A_\infty$-algebra structure on $H$ induced in Theorem 5.2.5 by this data, and let $(K, d_K)$ represent a trivial $A_\infty$-algebra.

Then there exists an isomorphism of $A_\infty$-algebras

$$\chi : (C, m') \xrightarrow{\sim} (H, m) \oplus (K, d_K)$$

such that

(i) $\chi_1 : C \xrightarrow{\sim} H \oplus K$ is the identity isomorphism of complexes.

(ii) The projection of $\chi$ onto $H$ is equal to the $A_\infty$-quasi-isomorphism $g : C \xrightarrow{\sim} H$ of Lemma 5.3.1.

(iii) The restriction to $H$ of the inverse isomorphism $\chi^{-1} : H \oplus K \rightarrow C$ is equal to the $A_\infty$-quasi-isomorphism $f : H \xrightarrow{\sim} C$ of Theorem 5.2.5.

**Proof.** This is the case of [CL17, Thm. 3.14] where the differential on $H$ is trivial. Moreover, an explicit formula for $\chi$ is given. □
5.4. The bar equivalence. In this section, we set up a dual formulation of $A_\infty$-algebras.

Recall that $\Sigma$ denotes the suspension operation on complexes $(A,d_A)$, so $(\Sigma A)^i = A^{i+1}$. We write $s$ for the canonical homogeneous degree $-1$ map $s : A \to \Sigma A$ and $\omega := s^{-1}$ for its inverse. Recall that $d_{\Sigma A} = -s \circ d_A \circ \omega$.

**Definition 5.4.1** (Bar construction). Let $A, A'$ be graded vector spaces. For $n \geq 1$, let $m_n : A^\otimes n \to A$ have homogeneous degree $2 - n$. Given this data, the bar construction consists of

- The homogenous degree 1 map $b_n : (\Sigma A)^\otimes n \to \Sigma A$ given by $b_n := -s \circ m_n \circ \omega^{\otimes n}$, for $n \geq 1$
- The coderivation $\bar{b} = b_A$ on the cofree cocomplete coalgebra
  \[\hat{T}_F^c(\Sigma A) := \bigoplus_{i \geq 0} (\Sigma A)^{\otimes i}\]
  determined by extending $\bigoplus_{n \geq 1} b_n : \hat{T}^c(\Sigma A) \to \Sigma A$ to a coderivation via the co-Leibniz rule.

We write $\text{Bar}(A)$ or $\text{Bar}(A,m)$ for the data $(\hat{T}_F^c(\Sigma A), b_A)$.

Likewise, we define the analogue of the bar construction for morphisms. Given the data $f = (f_n)_{n \geq 1}$ where $f_n : A^\otimes n \to A'$ has homogenous degree $1 - n$, we define

- the homogenous degree 0 map $g_n := (\Sigma A)^{\otimes n} \to \Sigma A'$ given by $g_n = s \circ f_n \circ \omega^{\otimes n}$, for $n \geq 1$
- the morphism $g^*$ of free graded cocomplete coalgebras $\hat{T}_F^c(\Sigma A) \to \hat{T}_F^c(\Sigma A')$ determined by extending $\bigoplus_{n \geq 1} g_n : \hat{T}_F^c(\Sigma A) \to \Sigma A'$ co-multiplicatively.

We write $\text{Bar}(f) : \text{Bar}(A') \to \text{Bar}(A)$ for $g^*$.

Note that we have not imposed any conditions on the graded linear maps $m_n$ (resp. $f_n$) in the construction above. The following theorem explains exactly when they satisfy $A_\infty$-compatibility conditions.

**Theorem 5.4.2.** Let $A, A'$ be graded vector spaces. For $n \geq 1$, let $m_n : A^\otimes n \to A$ and $m'_n : A'^{\otimes n} \to A'$ (resp. $f_n : A^\otimes n \to A'$) be linear of homogeneous degree $2 - n$ (resp. $1 - n$).

1. $(A,(m_n)_{n \geq 1})$ is an $A_\infty$-algebra if and only if the bar construction $(\hat{T}_F^c(\Sigma A), \bar{b})$ is a dg-algebra, i.e. $\bar{b}^2 = 0$.

2. Assuming that $(A,(m_n)_{n \geq 1})$ and $(A',m'_n)_{n \geq 1})$ are $A_\infty$-algebras, $f := (f_n)_{n \geq 1}$ is a morphism of $A_\infty$-algebras $f : A \to A'$ if and only if $g^* : \hat{T}_F^c(\Sigma A) \to \hat{T}_F^c(\Sigma A')$ is a morphism of dg-coalgebras, i.e. $g^*$ commutes with the coderivations $b'_n$ on $\hat{T}_F^c(\Sigma A)$ and $b_n$ on $\hat{T}_F^c(\Sigma A')$.

**Proof.** See e.g. [LV12, Lem. 9.2.2 and §9.2.11].

The following amplification of the theorem is also true.

**Corollary 5.4.3** (Bar equivalence). The bar construction defines a functor from $A_\infty$-algebras to the full subcategory dg-coalgebras determined by the cofree cocomplete objects. This functor is an isomorphism of categories.

**Proof.** The first statement follows directly from Theorem [5.4.2]. The second statement follows from the calculations of the bar construction (Definition [5.4.1]).
see that they do not lose any information and that a natural inverse to the bar
construction exists.

When the graded pieces \( A^n \subset A \) are finite-dimensional, it amounts to the same
thing to work with the dg-algebra dual to the dg-coalgebra \( \text{Bar}(A) \). This is the
dg-algebra that appears in \([2.1]\) We will use the notation therein for this complete
free dg-algebra:

**Definition 5.4.4.** Assume that \( A^n \) is finite-dimensional for all \( n \in \mathbb{Z} \). Then we
write

\[
\text{Bar}^*(A,m) := (\hat{T}_p \Sigma A^*, m^*, \pi)
\]

for the free complete dg-algebra that is graded-dual to the cofree cocomplete dg-
coalgebra \( \text{Bar}(A,m) \).

Note that we write \( m^* \) for the differential that is dual to the codifferential \( b \).

5.5. The Maurer-Cartan functor. We change notation: now \((A,d_A,m_{2,A})\) de-
notes a test object in the category of dg-algebras, while \((C,d_C,m_{2,C})\) denotes a
dg-algebra receiving an \( A_\infty \) quasi-isomorphism \( f : (H,m) \rightarrow (C,d_C,m_{2,C}) \) from a
minimal \( A_\infty \)-algebra \( H \). We follow \([CL11]\).

For simplicity, we assume that \( A \) is finite-dimensional and complete with max-
imal ideal \( m_A \), i.e. \( A \in A_{F}^{dg} \), which suffices for applications. We denote by \( A^* = (A^*,d_A^*,m_{2,A}^*) \) the natural dg-coalgebra dual to \( A \), with maximal ideal \( m_A^* \). Recall
that \( m_A \) is nilpotent.

We put an “\( A \)-linear” \( A_\infty \)-algebra structure \( m_A^* \) on \( H \otimes A \), defined by

\[
m_A^* := \text{id}_A \otimes m_n : A \otimes H^\otimes n \rightarrow A \otimes H.
\]

Likewise, we obtain \( b_n^A \).

**Definition 5.5.2.** Let \((H,m)\) be an \( A_\infty \)-algebra. Given a dg-algebra \( A \in A_{F}^{dg} \), a
Maurer-Cartan element for \( H \) valued in \( A \) is some \( \xi \in (m_A \otimes \Sigma H)^0 \) such that

\[
(d_A \otimes \text{id}_H)(\xi) + \sum_{n=1}^\infty b_n^A(\xi) = 0.
\]

Denote the set of such elements by \( \text{MC}(H,A) \). We usually write \( \xi \in (m_A \otimes H)^1 \),
but it is convenient to use its canonical image in \( (m_A \otimes \Sigma H)^0 \) so that we can forgo
signs that would appear in \((5.5.3)\).

**Remark 5.5.4.** For applications, we will restrict to classical complete algebras \( A \),
i.e. \( A \in A_{F} \). In this case, Maurer-Cartan elements come from \( m_A \otimes \Sigma (H^1) \).

**Remark 5.5.5.** In the case that an \( A_\infty \)-algebra \( C \) arises from a dg-algebra \((C,d_C,m_{2,C})\)
and \( A \in A_{F} \) is classical, the Maurer-Cartan equation takes on its classical formulation

\[
d_C(\xi) + m_{2,C}(\xi \otimes \xi) = 0.
\]

Sometimes the equation of Definition \((5.5.2)\) is called the homotopy Maurer-Cartan
equation to distinguish it from this case.

**Proposition 5.5.6.** Let \((H,m)\) be an \( A_\infty \)-algebra. The association

\[
A_{F}^{dg} \ni A \mapsto \text{MC}(H,A) \in \text{sets}
\]
is functorial in \( A \) and corepresentable by the bar construction of \( H \); that is,
\[
\text{MC}(H, A) = \text{Hom}_{dg}(A^*, \text{Bar}(H)).
\]
Moreover, if \( H^n \) is finite-dimensional for all \( n \in \mathbb{Z} \), then \( \text{MC}(H, A) \) is representable by the dual of the bar construction of \( H \), that is,
\[
\text{MC}(H, A) = \text{Hom}_{dg}(\text{Bar}^*(H), A).
\]

**Proof.** By calculation, e.g. [CLT11, Prop. 2.2]. \( \square \)

**Example 5.5.7.** Assume that \( H^n \) is finite-dimensional for all \( n \). Because of our interest in working with classical coefficient rings, we want to determine the classical complete \( F \)-algebra \( R \) such that for all classical augmented algebras \( A = (A, 0, m_2, A) \), we have
\[
\text{Hom}_{dg}(\text{Bar}^*(H), A) = \text{Hom}_{alg}(R, A).
\]
This \( R \) is the **classical hull** of \( \text{Bar}^*(H) = (\hat{T}(\Sigma H)^*, m^*, \pi) \), which we set up in §2.3.

### 5.6. The gauge action

Here we recall basic facts about the gauge action on Maurer-Cartan sets.

**Remark 5.6.1.** We emphasize that the version of the gauge action that we use makes sense in any characteristic. One reason we highlight this is that many accounts of the gauge action are expressed in the setting of Lie algebras, where the gauge action originated. There, the action is best expressed as an exponential and therefore only makes sense in characteristic zero. This action readily generalizes to the associative setting, but only some of the literature about this action have denominator-free expressions.

**Definition 5.6.2 (Unital associative gauge action).** Let \((C, d_C, m_C)\) be a unital dg-\( F \)-algebra. Let \( \beta \in C^1 \). Then the gauge action of \( \gamma \in (C^0)^\times \) on \( \beta \) is
\[
\gamma \cdot \beta := \gamma \beta \gamma^{-1} - d(\gamma) \gamma^{-1}.
\]

For non-unital dg-algebras, one can use the same formula to produce the following augmented gauge action.

**Definition 5.6.3 (Augmented associative gauge action).** Let \((C, d_C, m_C)\) be a non-unital complete dg-\( F \)-algebra. Then the gauge action of \( \gamma \in C^0 \) on \( \beta \in C^1 \) is the unital gauge action of \((1 - \gamma)^{-1} \), which is
\[
\gamma \cdot \beta := \beta - (1 - \gamma)^{-1}(\beta \gamma - \gamma \beta + d\gamma),
\]
where \((1 - \gamma)^{-1}\) is interpreted as the standard power series and \(1\) is interpreted as the identity action on \( C \).

**Proposition 5.6.4.** The gauge action in the expression above preserves the Maurer-Cartan subset of \( C^1 \) consisting of elements such that \( d\beta + \beta^2 = 0 \).

**Proof.** This is a straightforward calculation. \( \square \)

Recall that \( \text{MC}(C, A) \) amounts to the Maurer-Cartan elements in \((C \otimes m_A)^1\), when \((C, m)\) is an \( A_\infty \)-algebra over \( F \) (perhaps arising from a dg-algebra), \( A \) is a dg-
\( F \)-algebra, and \( A \) is complete. The \( A_\infty \)-algebra structure on \( C \otimes m_A \) is as in [5.5.1]. We have the Maurer-Cartan equation of [5.5.3]. In this setting, we formulate “strict” gauge equivalence in analogy with deformation theory: one conjugates by the multiplicative group \( 1 + M_d(m_A) \), as opposed to the entirety of \( M_d(A)^\times \).
Definition 5.6.5 (Strict gauge equivalence). Let $C$ be a dg-$F$-algebra and let $A \in \mathcal{A}_F^{dg}$. We say that $\beta, \beta' \in (C \otimes m_A)^1$ are strictly gauge equivalent when they are in the same orbit under the augmented gauge action of $(C \otimes m_A)^0$.

Let $(C, m)$ be an $A_\infty$-algebra, and let $A \in \mathcal{A}_F$. We say that $\beta, \beta' \in (C \otimes m_A)^1$ are gauge equivalent via $\gamma \in (C \otimes m_A)^0$ when

$$\beta - \beta' = (1 \otimes d_A)\gamma + \sum_{n=1}^{\infty} \sum_{j=1}^{n} (-1)^{j}m_n^A(\beta^{\otimes j-1} \otimes \gamma \otimes \beta'^{\otimes 1-j})$$

Denote by $MC(C, A)$ the set of strict equivalence classes of the Maurer-Cartan set $MC(C, A)$.

One may check that the $A_\infty$-version of strict gauge equivalence reduces to the dg-algebra case, when the $A_\infty$-algebra arises from a dg-algebra.

It is shown in [Pro11, Cor. 4.17] that gauge equivalence is an equivalence relation under a “connectedness” assumption. The point of this assumption is that certain expressions can be inverted, but since we presently work in the complete case (unlike loc. cit.), these assumptions can be dropped.

Remark 5.6.6. For an expression of an $A_\infty$ strict gauge action with denominators, see [Seg08, Defn. 2.9]. The denominator-free version above comes from [Pro11, Defn. 4.5], where it is expressed dually in $C$.

6. Associative deformations of a point

The goal of this section is to prove a non-commutative generalization of theorems of §3.1 — the determination of a deformation ring in terms of $A_\infty$-structure — about deformations of an absolutely irreducible representation.

6.1. Associative deformation theory. We use the conventions of §4.8 which set up the deformation theory of a representation

$$\rho : E \to \text{End}_F(V)$$

associative $F$-algebra $E$ with finite dimension $d := \dim_F V$. We especially use the coefficient categories $\mathcal{A}_F$ and $\mathcal{A}_F$ described there, consisting of objects $(A, m_A)$. When a $F$-basis for $V$ is chosen, we will write $\rho : E \to M_d(F)$.

Definition 6.1.1. Let $A \in \mathcal{A}_F$. A lift of $\rho$ over $A$ is a homomorphism $\rho_A : E \to \text{End}_F(V) \otimes A$ such that $\rho_A \otimes_A F = \rho$.

A deformation of $\rho$ over $A$ is an equivalence class of lifts $\rho_A : E \to \text{End}_F(V) \otimes A$ under the equivalence relation of conjugation (i.e. inner automorphism) by an element of $\text{End}_F(V) \otimes A$ whose reduction modulo $\text{End}_F(V) \otimes m_A$ is $\text{id}_V \in \text{End}_F(V)$ (any such element is a unit).

We define the lifting functor of $\rho$ (resp. the deformation functor of $\rho$), denoted $\text{Def}_\rho^{nc}$ (resp. $\text{Def}_\rho^{nc}$), as the functor from $\mathcal{A}_F$ to the category of sets sending $A$ to the set of lifts (resp. deformations) of $\rho$ over $A$.

To relate $\text{Def}_\rho^{nc}$ to homological invariants, we introduce the Hochschild cochain complex.
Definition 6.1.2. Let $E$ be an associative $\mathbb{F}$-algebra. Let $M$ be an $E$-bimodule. The Hochschild cochain complex, denoted $C^\bullet(E, M)$, is determined by the $E$-bimodules

$$C^\bullet(E, M) := \bigoplus_{i \geq 0} C^i(E, M), \quad C^i(E, M) := \text{Hom}_\mathbb{F}(E^\otimes i, M).$$

The differential $d = d^i : C^i(E, M) \to C^{i+1}(E, M)$ is determined by

$$d^i(f)(x_1, \ldots, x_{i+1}) = x_1 f(x_2, \ldots, x_{i+1}) + (-1)^i f(x_1, \ldots, x_i)x_{i+1}$$

$$+ \sum_{j=1}^{i} (-1)^j f(x_1, \ldots, x_j x_{j+1}, \ldots, x_{i+1}).$$

It is standard to verify that $d^{i+1} \circ d^i = 0$.

We denote by $H^\bullet(E, M)$ the cohomology graded vector space of $C^\bullet(E, M)$, which is called Hochschild cohomology.

Lemma 6.1.3. If $M$ has the structure of an associative $\mathbb{F}$-algebra, then the graded multiplication on $C^\bullet(E, M)$ induced by

$$C^i(E, M) \otimes C^j(E, M) \to C^{i+j}(E, M)$$

$$f \otimes g \mapsto [(x_1, \ldots, x_{i+j}) \mapsto f(x_1, \ldots, x_i) \cdot g(x_{i+1}, \ldots, x_{i+j})]$$

is a dg-algebra.

Proof. This is a standard computation. \qed

$\text{Def}_{\rho}^{nc, \square}$ has the following corepresentability. Indeed, the $E$-bimodule structure on $\text{End}_\mathbb{F}(\rho)$ is the natural one,

$$(x, y) \cdot f(v) = \rho(x) \cdot f(\rho(y) \cdot v) \quad \text{for } x, y \in E, v \in V, f \in \text{End}_\mathbb{F}(\rho).$$

For the statement of the proposition, we interpret $\beta \in C^1(E, \text{End}_\mathbb{F}(\rho)) \otimes \mathfrak{m}_A$ as a function $\beta : E \to \text{End}_\mathbb{F}(\rho) \otimes \mathfrak{m}_A$.

Theorem 6.1.5. We have the dg-$\mathbb{F}$-algebra $C = C^\bullet(E, \text{End}_\mathbb{F}(\rho))$. For $A \in \mathcal{A}_\mathbb{F}$, there is the following natural bijection between $A$-valued Maurer-Cartan elements for $C$ and lifts of $\rho$ to $A$. That is, $\text{MC}(C, A) \cong \text{Def}_{\rho}^{nc, \square}(A)$ via

$$\xi \mapsto (\rho \circ \xi : E \to \text{End}_\mathbb{F}(V) \otimes A).$$

In particular, $\text{Bar}(C)$ corepresents $\text{Def}_{\rho}^{nc, \square}$.

Proof. Using Definition 5.5.2 and Remark 5.5.5 we calculate that an element

$$\xi \in \mathfrak{m}_A \otimes C^1(E, \text{End}_\mathbb{F}(\rho)) \cong \text{Hom}_\mathbb{F}(E, \text{End}_\mathbb{F}(\rho) \otimes \mathfrak{m}_A)$$

is Maurer-Cartan if and only if it obeys the relation

$$\xi(x_1 \cdot x_2) = \xi(x_1) \cdot \xi(x_2) + \rho(x_1) \cdot \xi(x_2) + \xi(x_1) \cdot \rho(x_2).$$

From this, one readily observes that an element of $\text{Hom}_\mathbb{F}(E, \text{End}_\mathbb{F}(V) \otimes \mathfrak{m}_A)$ is Maurer-Cartan if and only if

$$\rho \circ \xi : E \to \text{End}_\mathbb{F}(V) \otimes A \cong (\text{End}_\mathbb{F}(V) \oplus \text{End}_\mathbb{F}(V) \otimes \mathfrak{m}_A)$$

is a homomorphism.

Now the corepresentability claim follows from Proposition 5.5.6. \qed
We now turn to deformations. We write
\[(6.1.6) \quad E_\rho^\wedge := \lim_{\xi \to 1} E / \ker(\rho)^i.\]
First, we notice that the natural map \(E \to E_{\rho}^\wedge\) is a deformation of \(\rho\). Indeed, because all deformations of matrix algebras are known to be trivial (or apply \cite{Lau02} Thm. 1.2), we get that \(E / \ker(\rho)^i\) is isomorphic to a matrix algebra \(M_{\delta}(A_i)\) for some \(A_i \in \mathcal{A}\). In the limit, we choose an isomorphism
\[(6.1.7) \quad E_\rho^\wedge \simeq M_{\delta}(R_{\rho}^{nc}) \simeq \text{End}_\mathbb{F}(V) \otimes R_{\rho}^{nc}\]
for a chosen \(R_{\rho}^{nc} \in \mathcal{A}\), well-defined up to inner automorphism. This choice is a lift of \(\rho\) to \(R_{\rho}^{nc}\). So it is fair to call \(E \to E_{\rho}^\wedge\) is a deformation valued in \(R_{\rho}^{nc}\).

The isomorphism \[(6.1.7)\] realizes a Morita equivalence between \(E_{\rho}^\wedge\) and \(R_{\rho}^{nc}\), as follows. Selecting the idempotent \(e_{11} \in M_{\delta}(R_{\rho}^{nc}) \simeq E_{\rho}^\wedge\) (the matrix with 1 concentrated in the \((1,1)\)-coordinate) via isomorphisms above, the Morita equivalence of categories is explicitly given by
\[(6.1.8) \quad \begin{array}{c}
E_{\rho}^\wedge \text{-Mod} \xrightarrow{\sim} R_{\rho}^{nc} \text{-Mod} \\
W \mapsto e_{11} W \\
V \otimes Y \leftrightarrow Y
\end{array}\]

Now the representability of \(\text{Def}_{\rho}^{nc}\) follows from the explicit Morita equivalence.

**Proposition 6.1.9.** Assume that \(\rho : E \to \text{End}_\mathbb{F}(V)\) is absolutely irreducible and let \(R_{\rho}^{nc} \in \mathcal{A}\) as above. Then \(\text{Def}_{\rho}^{nc}\) is isomorphic to the inner automorphism quotient of the Hom-functor on \(\mathcal{A}\) of \(R_{\rho}^{nc}\). That is, we have a functorial isomorphism
\[\text{Def}_{\rho}^{nc}(A) \xrightarrow{\sim} \text{Hom}_\mathbb{F}(R_{\rho}^{nc}, A)/\sim_A,\]
where \(\sim_A\) indicates the equivalence relation of inner \(\mathbb{F}\)-algebra automorphisms of \(A\). This isomorphism is given by applying \((-) \otimes \text{End}_\mathbb{F}(V)\) to (the domain and codomain of) a representative \(\eta : R_{\rho}^{nc} \to A\) of an element of \(\text{Hom}_\mathbb{F}(R_{\rho}^{nc}, A)/\sim_A\).

**Remark 6.1.10.** We see that when we restrict the test coefficients of \(\text{Def}_{\rho}^{nc}\) from \(\mathcal{A}\) to \(\mathcal{C}\), which we call \(\text{Def}_{\rho}\), the abelianization of \(R_{\rho}^{nc}\) represents \(\text{Def}_{\rho}\).

Finally, we prove that \(\text{Def}_{\rho}^{nc}\) is representable by \(C\) when taken up to strict gauge equivalence. The salient point is that strict gauge equivalence amounts to strict isomorphism.

**Proposition 6.1.11.** We have the dg-\(\mathbb{F}\)-algebra \(C = C^\bullet(E, \text{End}_\mathbb{F}(\rho))\). For \(A \in \mathcal{A}\), there is the following natural bijection between \(A\)-valued Maurer-Cartan elements for \(C\) and lifts of \(\rho\) to \(A\). That is, \(MC(C, A) \xrightarrow{\sim} \text{Def}_{\rho}^{nc}(A)\).

**Proof.** In light of Theorem 6.1.5 it remains to prove that the strict gauge action of \(\gamma \in C^0 \otimes m_A\) on \(\xi \in MC(C, A)\) amounts to conjugation of \(\rho \oplus \xi\) by \((1 - \gamma)\). Indeed, we calculate the conjugation
\[
(1 - \gamma)^{-1}(\rho \oplus \xi)(1 - \gamma) = \sum_{i=0}^{\infty} \gamma^i(\rho + \xi - \rho\gamma - \xi\gamma)
= \rho + \xi + \sum_{i=0}^{\infty} \gamma^i(\xi\gamma - \xi - d\gamma)
= \rho \oplus \xi',
\]
where
\[ \xi' := \xi - (1 - \gamma)^{-1}([\xi, \gamma] + d\gamma). \]
This is exactly the strict gauge action of Definition 5.6.3. □

6.2. **Associative deformation theory determined by \( A_\infty \)-structure.** We aim for an analogue of Theorem 6.1.5 giving a homological expression for \( \text{Def}_{\rho}^\text{nec} \). This analogue will be formulated in terms of an \( A_\infty \)-structure on Hochschild cohomology. We continue with the objects defined above: \( E, \rho \), and \( C \). Let
\[ H := H^*(C) = H^*(E, \text{End}_F(\rho)) \]
denote Hochschild cohomology of \( \text{End}_F(\rho) \).

We choose a homotopy retract structure on \((H, 0)\) relative to \((C, d_C)\) as in Example 5.2.2 and apply the results of §5, producing
- a minimal \( A_\infty \)-structure on \( H = H^*(G, \text{End}_F(\rho)) \), denoted \( (H, m) = (H, (m_n)_{n \geq 2}) \), extending its canonical graded algebra structure \( m_2 \).
- a quasi-isomorphism \( f : H \to C \) of \( A_\infty \)-algebras (Corollary 5.2.6) and
- an isomorphism \( \chi : C \to H \oplus K \) of \( A_\infty \)-algebras, where the projection to \( H \) is a left inverse and right quasi-inverse to \( f \) (Theorem 5.3.3), and \( (K, d_K) \) is a trivial \( A_\infty \)-algebra.

We assemble the following two facts.

**Lemma 6.2.1.** For a trivial \( A_\infty \)-algebra \( (K, d_K) \), one has \( \text{MC}(K, -) = * \) and
\[ \text{MC}(K, A) \cong B^1 \otimes m_A \]
for \( A \in A_F \). For the particular trivial \( A_\infty \)-algebra \( (K, d_K) \) produced from \( C = C^*(E, \text{End}_F(\rho)) \), along with a homotopy retract on \( C \) given in Example 5.2.2, we have
\[ \text{MC}(K, A) \cong \frac{\text{End}_F(\rho)}{\text{diag}(F)} \otimes m_A, \]
where \( \text{diag}(F) \) denotes the scalar matrix subfield \( F \hookrightarrow \text{End}_F(\rho) \).

**Proof.** Firstly, recall that \( m_{K,1} : B^i \oplus L^i \to B^{i+1} \oplus L^{i+1} \) arises by restriction of the differential \( d_{\nabla} : C^i \to C^{i+1} \). It is the zero map on \( B^i \) along with the isomorphism \( L^i \cong B^{i+1} \). Considering the case \( i = 1 \) and recalling that \( m_{K,n} = 0 \) for \( n \geq 2 \), we have the calculation of \( \text{MC}(K, A) \). Considering the case \( i = 0 \), we see that the gauge action is a torsor, hence \( \text{MC}(K, -) \) is a point. The final claim follows from the canonical isomorphism \( B^1 \cong C^0 / \ker(d_0^C) \), noting that \( C^0 \cong \text{End}_F(\rho) \) and \( d_0^C \) kills exactly the scalar matrices. □

**Lemma 6.2.2.** Assume that \( H^i \) is finite-dimensional for all \( i \in \mathbb{Z} \), so that \( \text{MC}(H, -) \) is pro-represented on \( A_F \) by the classical hull \( R \) of \( \text{Bar}^*(H) \in A_F^{dg} \) (see Example 5.5.7). For \( A \in A_F \), this pro-representability maps strict equivalence classes in \( \text{MC}(H, A) \) isomorphically onto inner automorphism classes in \( \text{Hom}_F(R, A) \).

The key point is that \( H \) is connected, in the sense that \( H^0 \cong F \), arising from the center of \( C \).
Proof. Let $\gamma \in H^0 \otimes m_A$ and let $\beta, \beta' \in H^1 \otimes m_A$. Because $H^0 \cong \mathbb{F}$ and it arises from the center of $C^0$, the higher multiplications are trivial on $H^0$. That is, when $\gamma' \in H^0$ and $\delta_i \in H^1$ for $1 \leq i < n$, then $m_n(\gamma' \otimes \delta_1 \otimes \cdots \otimes \delta_{n-1}) = 0$ for $n = 1$ and $n \geq 3$, and for any other tensor-permutation of the arguments. Therefore an $A_\infty$-strict gauge equivalence (Definition 5.6.5) between $\beta$ and $\beta'$ via $\gamma$ reduces to

$\beta - \beta' = -\gamma \beta' + \beta \gamma$, hence $\beta(1 - \gamma) = (1 - \gamma) \beta'$

(where we are implicitly using $m^2_A$ as multiplication). This is the relation of conjugation by $(1 - \gamma)$, and only the $A$-part of the conjugation by $H^0 \otimes m_A$ is non-trivial. □

With these two lemmas in place, the idea is to use the isomorphism of Maurer-Cartan sets and their compatible gauge relations under the $A_\infty$-isomorphism $\chi : C \cong H \oplus K$.

**Theorem 6.2.3.** Let $E, \rho, C, H$ be as above. Choose a homotopy retract structure on $(H,0)$ relative to $(C,d_C)$ as in Example 5.2.2, which gives the additional data $(H,m)$, $f$, and $\chi$. For $A \in A_\mathbb{F}$, this choice determines isomorphisms

$$\text{MC}(H,A) \xrightarrow{f_*} \text{MC}(C,A) \xrightarrow{\text{Def}_\rho} \text{Def}_\rho(A)$$

and they are functorial in $A$.

In words, the theorem states that $\text{Def}_\rho$ is corepresented by the $A_\infty$-algebra $(H,m)$ up to gauge equivalence. We also see that for Maurer-Cartan elements of $H$ in $A$, gauge equivalence amounts to inner automorphism in $A$. We write $f_*$ in the statement to indicate the map on gauge equivalence classes induced from the map of Maurer-Cartan sets

$$f_* : \text{MC}(H,-) \to \text{MC}(C,-) \cong \text{Def}_\rho(\text{-})$$

induced by $f : H \to C$.

**Proof.** By Theorem 5.3.3, we have available an $A_\infty$-isomorphism

$$\chi : (C,d_C,m_{2,C}) \xrightarrow{\sim} (H,m) \oplus (K,d_K).$$

Clearly this induces isomorphisms of Maurer-Cartan functors

$$\text{MC}(C,-) \xrightarrow{\sim} \text{MC}(H \oplus K,-), \quad \text{MC}(C,-) \xrightarrow{\sim} \text{MC}(H \oplus K,-).$$

Combining Proposition 6.1.11 with Lemmas 6.2.1 and 6.2.2, noting that gauge equivalence decomposes along the decomposition $H \oplus K$, the claim follows immediately. □

**Remark 6.2.4.** We see in the statement of Theorem 6.2.3 an instance of “the homotopy invariance of the Maurer-Cartan functor”: after gauge equivalence, it is a quasi-isomorphism invariant of $A_\infty$-algebras. This is well-known but rarely stated in arbitrary characteristic. In this generality, it can be derived from Theorem 5.3.3 from [CL17], Lemma 6.2.1, and a generalization of Lemma 6.2.2 for minimal $A_\infty$-algebras that we do not require here.

We are interested in amplifying Theorem 6.2.3 to give a cohomological presentation for $R^\rho$ and explicit formulas for representations associated to elements of $\text{MC}(H,A)$. The data $f, \chi$ induced by the homotopy retract is suited for this.
Under the assumption that $H^i$ is finite-dimensional for all $i \geq 0$, recall that

$$R := \frac{T_{\mathbb{F}}((\Sigma H^1)^*)}{(m^*((\Sigma H^2)^*))} \in \hat{A}_{\mathbb{F}}$$

denote the classical hull of the dg-algebra $\text{Bar}^*(H)$ set up in Example 5.5.7.

**Corollary 6.2.6.** Let $E, \rho, C, H$ be as above. Choose a homotopy retract structure on $(H, 0)$ relative to $(C, d_C)$ as in Example 5.2.2, which gives the additional data $(H,m)$, $f$, and $\chi$. Under the additional assumption that $H^i$ is finite-dimensional for all $i \geq 0$, these data

1. determine an isomorphism $\rho^{\mathfrak{u}} : E^\wedge_{\rho} \xrightarrow{\sim} \text{End}_{\mathbb{F}}(V) \otimes R$

   given by, for $x \in E$,

   $$x \mapsto \rho(x) + \sum_{i=1}^{\infty} (\varepsilon \mapsto (f_i(\varepsilon))(x)) \in \text{End}_{\mathbb{F}}(V) \otimes R$$

   where $\varepsilon$ is a generic element of $(\Sigma H^1)^{\otimes i} = \Sigma H^1(E, \text{End}_{\mathbb{F}}(V))^{\otimes i}$.

2. Upon the additional choice of an idempotent $e^{11} \in E^\wedge_{\rho}$ used to define $R^{\text{nc}}_{\rho}$, $\rho^{\mathfrak{u}}$ induces an isomorphism $R^{\text{nc}}_{\rho} \xrightarrow{\sim} R$.

We give an explanation of the notation $(\varepsilon \mapsto (f_i(\varepsilon))(x))$. By definition of $f = (f_n)_{n \geq 1}$, we find $f_i(\varepsilon) \in C^i(E, \text{End}_{\mathbb{F}}(V)) = \text{Hom}_{\mathbb{F}}(E, \text{End}_{\mathbb{F}}(V))$, which is a function that can be evaluated on $x \in E$. So, altogether, $(\varepsilon \mapsto (f_i(\varepsilon))(x))$ is an element of $(\Sigma H^1(E, \text{End}_{\mathbb{F}}(V))^*)^{\otimes i} \otimes \text{End}_{\mathbb{F}}(V)$. This determines an element of $\text{End}_{\mathbb{F}}(V) \otimes R$ via the surjection $T((\Sigma H^1)^*) \twoheadrightarrow R$ of (6.2.5).

**Remark 6.2.7.** Theorem 3.1.1 follows more-or-less directly from Corollary 6.2.6.

**Proof.** First we produce $\rho^{\mathfrak{u}}$ and justifies its formula. We claim that $\rho^{\mathfrak{u}}$ is the $R$-valued lift of $\rho$ arising from the map $\text{Bar}(f) : \text{Bar}(H) \to \text{Bar}(C)$. Indeed, recall from Theorem 6.1.3 that $\text{Bar}(C)$ corepresents the lifting functor $\text{Def}^{\text{nc}}_{\rho}$ on Artinian augmented F-algebras. Then recall that $R$ is a limit of such algebras, and it is also the classical hull of the dual dg-algebra to $\text{Bar}(H)$. To prove the claim, note that the sum of $f_i : (H^1)^{\otimes i} \to C^1$ over $i \geq 1$ determines an element of

$$\prod_{i \geq 1} ((\Sigma H^1)^* \otimes C^1).$$

This element reduces to the Maurer-Cartan element $\xi_R$ in $C^1 \otimes m_R$ arising from $\text{Bar}(H) \to \text{Bar}(C)$. Finally, as in the proof of Theorem 6.1.5, $\xi_R$ is a F-linear map from $E$ to $\text{End}_{\mathbb{F}}(V) \otimes m_R$ that determines a homomorphism $\rho \otimes \xi_R : E \to \text{End}_{\mathbb{F}}(V) \otimes R$ that appears in the formula in (1). We observe that its codomain is local and complete, so $\rho \otimes \xi_R$ induces the map $\rho^{\mathfrak{u}}$.

Now that we know that $\rho^{\mathfrak{u}}$ is a homomorphism, we produce

$$e\rho^{\mathfrak{u}}e : R^{\text{nc}}_{\rho} = eE^\wedge_{\rho}e \longrightarrow R \cong \rho^{\mathfrak{u}}(e)(\text{End}_{\mathbb{F}}(V) \otimes R)\rho^{\mathfrak{u}}(e).$$

The following collection of facts implies that $e\rho^{\mathfrak{u}}e$ is an isomorphism. Firstly, note that $R$ pro-represents $\text{MC}(H, -)$ on $\hat{A}_{\mathbb{F}}$ by its definition. Next, Theorem 6.2.3 draws an isomorphism $\text{MC}(H, -) \xrightarrow{\sim} \text{Def}^{\text{nc}}_{\rho}$. Proposition 6.1.9 shows that $R^{\text{nc}}_{\rho}$ represents $\text{Def}^{\text{nc}}_{\rho}$ after inner automorphism of the coefficients. Lemma 6.2.2 shows that the
projection $\text{MC}(H,-) \to \text{MC}(H,-)$ amounts to inner automorphism classes in the coefficients in $A_r$. Therefore $R^\nu$ and $R$ pro-represent the functors on $A_r$ that are identified via the map $\hat{f}_\nu$ of Theorem 6.2.3 up to inner automorphism. Because the formula for $\rho^\nu$ realizes the map $f_\nu$ discussed after Theorem 6.2.3 we see that $e\rho^\nu e$ is compatible with this isomorphism of functors, up to inner automorphism. Therefore $e\rho^\nu e$ is itself an isomorphism. □

7. Associative deformations of multiple points

The goal of this section is to generalize the results of [6] to the case where we are deforming multiple points. Stated representation-theoretically, we are deforming a semi-simple representation with distinct simple summands. We will follow the approach of [Seg08] in order to study this problem with multiple-pointed coefficients: see §1.3 and §2 of loc. cit. We carry out the work to combine [6] with the content of [Seg08] §2.

7.1. Setting up the data for multiple points. We adapt the notation established at the outset of §6.1. Now $\rho : E \to \text{End}_F(V)$ is a semi-simple representation on a $F$-vector space $V$. We write $\rho \cong \bigoplus_{i=1}^r \rho_i$, where $\rho_i : E \to \text{End}_F(V_i)$ and we fix an isomorphism $V \cong \bigoplus_{i=1}^r V_i$. We assume that the summands $\rho_i$ are absolutely irreducible and pairwise non-isomorphic.

As in [Seg08] §1.3], we use coefficient algebras on $r$ points. We write $F^r$ for the $r$-times product algebra $F \times \cdots \times F$. As a convention, we use the term “$F^r$-bimodule” as a shortening of the standard terminology “$(F^r,F^r)$-bimodule.”

**Definition 7.1.1.** For $1 \leq i \leq r$, we write $1_i \in F^r$ for the element with 1 concentrated in the $i$-th coordinate. For a $F^r$-bimodule $M$ and $1 \leq i, j \leq r$, we write $A_{ij}$ for $1_i \cdot A \cdot 1_j$, so that $M \cong \bigoplus_{i,j} A_{ij}$.

Let $\text{Alg}_F^r$ denote the category of $F^r$-algebras, that is, associative unital algebra objects in the category of $F^r$-bimodules. We write $F^r \in \text{Alg}_F^r$ for the standard $\text{Alg}_F^r$ structure on the ring $F^r$, i.e. the identity maps $F^r \to F^r$ are the structure maps. Then we observe that $F^r$ is a unit for the symmetric monoidal (tensor) product $\otimes$ in $\text{Alg}_F^r$ by assigning to $A, A' \in \text{Alg}_F^r$ the vector space

$$(A \otimes A')_{ij} := A_{ij} \otimes A'_{ij},$$

with coordinate-wise multiplication. Note that $\otimes$ is the underlying tensor product in the category of $F^r$-bimodules. In contrast, given $F^r$-bimodules $M, N$, we use $M \otimes_{F^r} N$ to denote the usual tensor product, using the right $F^r$-module structure on $M$ and the left $F^r$-module structure on $N$ to produce a $F^r$-bimodule. Finally, the undecorated symbol “$\otimes$” is understood to be over $F$, as usual.

An augmentation of $A \in \text{Alg}_F^r$ is a morphism $A \to F^r$ in $\text{Alg}_F^r$. Let $A_F^r$ denote the category of augmented $F^r$-algebras that have finite $F$-dimension. We write $m_A \subset A$ for the augmentation ideal of $A \in A_F^r$, so $A/m_A \cong F^r$.

Given a $F^r$-bimodule $M$, we have the completed tensor product $F^r$-algebra

$$\hat{T}_{F^r} M := \prod_{i \geq 0} M^\otimes_{F^r}.$$
We understand \( \text{End}_F(V) \) to be a \( F \)-algebra by sending \( 1_i \) to the projection operator from \( V \) to \( V_i \). We will also use the \( F \)-subalgebra of \( \text{End}_F(V) \),

\[
\text{End}_F(V) := \bigoplus_{i=1}^r 1_i \text{End}_F(V) 1_i \cong \bigoplus_{i=1}^r \text{End}_F(V_i).
\]

**Warning 7.1.2.** Note that the present notion of \( F \)-algebra is not the same as “an associative ring receiving a homomorphism from \( F \) to its center.” An \( F \)-algebra does receive a canonical homomorphism from \( F \), but it is not central. See [Seg08, §1.3] for equivalent formulations of \( \text{Alg}_F \). Most useful here is the following formulation: an associative \( F \)-algebra with an ordered complete set of orthogonal idempotents.

### 7.2. \( \text{dg} \)-algebras, \( \text{A}_\infty \)-algebras, and representability over multiple points.

In Theorem 6.1.5, we found a bijection between associative lifts and Maurer-Cartan elements for the Hochschild complex. We have the following \( r \)-pointed generalizations of the objects.

**7.2.1. \( \text{dg} \)-algebras.** A \( \text{dg} \)-\( F \)-algebra amounts to a \( F \)-linear \( \text{dg} \)-category on \( r \) objects (labeled by \( \{1, \ldots, r\} \)), or, equivalently, additional \( r \)-pointed structure on a \( \text{dg} \)-algebra over \( F \). It will suffice to consider the example we are concerned with: morphisms in this category are the Hochschild cochain complexes

\[
\text{Hom}(j, i) := C^\bullet(E, \text{Hom}_F(\rho_j, \rho_i)) \quad \text{for } 1 \leq i, j \leq r
\]

(the \( E \)-bimodule structure of \( \text{Hom}_F(\rho_j, \rho_i) \) is just like (6.1.4)), where the composition of morphisms arises from

\[
\text{Hom}_F(\rho_k, \rho_j) \otimes \text{Hom}_F(\rho_j, \rho_i) \to \text{Hom}_F(\rho_k, \rho_i).
\]

This composition is compatible with the Hochschild differential for the same reason as Lemma 6.1.3. Indeed, it follows from applying the statement of Lemma 6.1.3 (verbatim) to the Hochschild complex of \( \text{End}_F(\rho) \), and then using its \( F \)-algebra structure to deduce the compatibility for the \( \text{dg} \)-category.

We leave the notion of morphisms to the reader.

**7.2.2. \( \text{A}_\infty \)-algebras.** Similarly to \( \text{dg} \)-algebras, we may view an \( \text{A}_\infty \)-\( F \)-algebra as a \( F \)-linear \( \text{A}_\infty \)-category on \( r \) objects. That is, \( \text{Hom}(i, j) \) is a complex with differential \( m_1 \), and for \( n \geq 2 \) and any finite sequence \( i_0, \ldots, i_n \) in \( \{1, \ldots, r\} \), there is a \( F \)-linear composition law \( m_n \) on

\[
m_n : \text{Hom}(i_0, i_1) \otimes \cdots \otimes \text{Hom}(i_{n-1}, i_n) \to \text{Hom}(i_0, i_n) \quad \text{of degree } 2 - n.
\]

The \( m = (m_n)_{n \geq 1} \) are required to satisfy the compatibility conditions of (5.1.2). We will mainly discuss \( \text{A}_\infty \)-\( F \)-algebra structures on \( H^\bullet(E, \text{End}_F(V)) \); namely,

\[
\text{Hom}(i, j) = H^\bullet(E, \text{End}_F(\rho_j, \rho_i)).
\]

The composition \( m_n \) on \( H^\bullet(E, \text{End}_F(V)) \) is a sum of maps of the form (7.2.1) by applying the direct sum decomposition \( \text{End}_F(\rho) \cong \bigoplus_{1 \leq i, j \leq r} \text{End}_F(\rho_j, \rho_i) \).

We leave the notion of morphisms to the reader.
7.2.3. **Bar construction.** The bar construction involves taking linear duals and suspensions, all of which naturally respects $F^r$-structure. The bar equivalence of Corollary 5.4.3 also generalizes, giving an isomorphism of categories between $A_\infty$-$F^r$-algebras and cofree cocomplete (over $F^r$, i.e. coaugmented over $F^r$) co-dg-$F^r$-algebras.

The cofree cocomplete co-dg-$F^r$-algebra corresponding to an $A_\infty$-$F^r$-algebra $(H, m)$ is the data of a codifferential on $\hat{T}_{F^r} \Sigma H \cong \bigoplus_{i \geq 0} \Sigma H^{\otimes r^i}$. When $H^i$ has finite $F^r$-dimension for all $i \in \mathbb{Z}$, then we form the dual dg-algebra $(\hat{T}_{F^r} \Sigma H^*, m^*, \pi)$.

7.2.4. **Maurer-Cartan functor.** For an $A_\infty$-$F^r$-algebra $(H, (m_n)_{n \geq 1})$ and any $A \in A_\infty$, a Maurer-Cartan element for $H$ valued in $A$ is some $\xi \in (m_A \otimes \Sigma H)^0$ such that the Maurer-Cartan equation (5.5.3) holds. The functor of Maurer-Cartan elements is corepresentable over $F^r$, in direct analogy to Proposition 5.5.6.

7.2.5. **Kadeishvili’s theorem and the decomposition theorem.** Next, in order to discuss deformations, we need an $r$-pointed version of Kadeishvili’s theorem (Corollary 5.2.6). The key point is that the homotopy retract relating the complex $C$ and its cohomology $H$ should respect $F^r$-bimodule structure. To make this clear, we state the $r$-pointed generalization of Definition 5.2.1.

**Definition 7.2.2.** Let $(A, d_A)$, $(C, d_C)$ be complexes of $F^r$-bimodules. We call $(A, d_A)$ a $r$-pointed homotopy retract of $(C, d_C)$ when they are equipped with maps

$$h : C \rightarrow A$$

such that $p$ and $i$ are morphisms of complexes of $F^r$-bimodules, $h : C \rightarrow C[1]$ is a morphism of graded $F^r$-bimodules, $i_{d_C} - ip = d_Ch + hd_C$, and $i$ is a quasi-isomorphism of complexes of $F^r$-bimodules.

Once this is done, Corollary 5.2.6 and Theorem 5.3.3 apply to $A_\infty$-$F^r$-algebras.

**Example 7.2.3.** To illustrate this for the dg-$F^r$-algebra $C = C^*(E, \text{End}_F(V))$ and its cohomology $H$, the point is that the retract datum

$$i : (H, 0) \rightarrow (C, d_C)$$

must lift cohomology classes in $H^*(E, \text{Hom}_F(\rho_j, \rho_i))$ to a cochain in the $F$-subspace $C^*(E, \text{Hom}_F(\rho_j, \rho_i)) \subset C^*(E, \text{End}_F(V))$.

Then one may readily deduce from Example 5.2.7 or 5.2.8 that the resulting

- $A_\infty$-structure $m$ on $H$,
- $A_\infty$-quasi-isomorphism $f : H \rightarrow C$, and
- $A_\infty$-isomorphism $\chi : C \rightarrow H \oplus K$

respect $F^r$-structure, using the formulas for $m$, $f$, and $\chi$.

7.3. **Deformation theory of $r$ points.** We begin by setting up an an $r$-pointed version of the 1-pointed lifting functor $\text{Def}^{nc, \square}$ and 1-pointed deformation functor $\text{Def}^{nc}_\rho$ that were defined in Definition 6.1.1.
Remark 7.3.1. We comment on the appropriate notion of $r$-pointed notion of strict equivalence: there are two possible options, and we will show that they are equivalent (Proposition 7.3.5) when applied to the appropriate notions of lift. At the least, conjugation of a lift $\rho_A$ of $\rho$ should preserve the lifting property. The largest subgroup of $(\text{End}_E(V)\otimes A)^\times$ that does this is $F^r + \text{End}_E(V)\otimes m_A$, where $F^r \hookrightarrow \text{End}_E(V)$ arises from its $F^r$-algebra structure. Within this subgroup, we can also insist on preserving $F^r$-structure when conjugating, i.e. we demand an inner automorphism of $\text{End}_E(V)\otimes A$ as an $F^r$-algebra. This subgroup is
\[ F^r + \text{End}_E(V)\otimes m_A \subset F^r + \text{End}_E(V)\otimes m_A. \]

The smaller one is more naturally $r$-pointed. However, we are forced to use the larger relation because $E$ has no natural $r$-pointed structure.

Definition 7.3.2. Let $A \in \mathcal{A}_E^r$. A lift of $\rho$ over $A$ is a $\mathcal{A}$-algebra homomorphism $\rho_A : E \rightarrow \text{End}_E(V)\otimes A$ such that $\rho_A\otimes_A F^r = \rho$.

A deformation of $\rho$ over $A$ is an equivalence class of lifts $\rho_A : E \rightarrow \text{End}_E(V)\otimes A$ under the equivalence relation of conjugation by $F^r + \text{End}_E(V)\otimes m_A$.

We define the lifting functor of $\rho$ (resp. the deformation functor of $\rho$) on $\mathcal{A}_E^r$, denoted $\text{Def}^\rho_{\mathcal{A}, r}$ (resp. $\text{Def}^\rho_{\mathcal{A}, r}$), as the functor from $\mathcal{A}_E^r$ to the category of sets sending $A$ to the set of lifts (resp. deformations) of $\rho$ over $A$.

To produce $r$-pointed notions of lift and deformation, we begin with $E \rightarrow E^\wedge_r$, defined in (6.1.6).

Let $\tilde{e}_i \in \text{End}_E(V_i)$ be a projection operator onto the $i$-dimensional subspace of $V_i$, and let $e = \sum_i \tilde{e}_i \in \text{End}_E(V)$. Choose orthogonal idempotent lifts $e_i \in E^\wedge_r$ of $\tilde{e}_i$ via $\rho_i : E^\wedge_r \rightarrow \text{End}_E(V_i)$. Letting $e = \sum_{i=1}^r e_i$, we get a Morita equivalence between $E^\wedge_r$ and
\[ R^\rho_{nc,r} := e E^\wedge_r e \in \mathcal{A}_E^r, \]
where the $F^r$-algebra structure on $R^\rho_{nc,r}$ is determined by $(e_i)_i$. The inverse equivalence on algebras is realized by
\[
E^\wedge_r \simeq (M_{d_i \times d_j}((R^\rho_{nc,r})_{i,j}))_{i,j} \simeq \text{End}_E(V)\otimes R^\rho_{nc,r};
\]
for a proof of this, apply [Lau02, Thm. 1.2] in the limit on $E/\ker(\rho)^i$.

Now that we have a choice (7.3.3) of $F^r$-algebra structure on $E^\wedge_r$, we can set up $r$-pointed lifting and deformation functors.

Definition 7.3.4. Let $A \in \mathcal{A}_E^r$. A $r$-lift of $\rho$ over $A$ is a $\mathcal{A}$-algebra homomorphism $\rho_A : E^\wedge_r \rightarrow \text{End}_E(V)\otimes A$ such that $\rho_A\otimes_A F^r = \rho$.

A $r$-deformation of $\rho$ over $A$ is an equivalence class of $r$-lifts $\rho_A : E^\wedge_r \rightarrow \text{End}_E(V)\otimes A$ under the equivalence relation of conjugation by $F^r + \text{End}_E(V)\otimes m_A$.

We define the lifting functor of $\rho$ (resp. the deformation functor of $\rho$) on $\mathcal{A}_E^r$, denoted $\text{Def}^\rho_{\mathcal{A}, r}$ (resp. $\text{Def}^\rho_{\mathcal{A}, r}$), as the functor from $\mathcal{A}_E^r$ to the category of sets sending $A$ to the set of lifts (resp. deformations) of $\rho$ over $A$.

While there is clearly a natural proper inclusion of lifting functors on $\mathcal{A}_E^r$
\[ \text{Def}^\rho_{\mathcal{A}, r} \hookrightarrow \text{Def}^\rho_{\mathcal{A}, r}, \]
the two notions of deformation are equivalent.

Proposition 7.3.5 (Segal). There is a natural isomorphism of functors on $\mathcal{A}_E^r$
\[ \text{Def}^\rho_{\mathcal{A}, r} \sim \text{Def}^\rho_{\mathcal{A}, r}. \]
Proof. This is [Seg08, Prop. 2.11 and Lem. 2.12].

Having set up the deformation theory, we examine what arises from the natural $r$-pointed structures on the Hochschild cochain dg-$\mathcal{F}^r$-algebra $C = C^\bullet(E, \text{End}_F(V))$. We immediately find the following $r$-pointed version of Theorem 6.1.5, where the left isomorphism is simply the representability of the Maurer-Cartan functor.

**Theorem 7.3.6.** Let $A \in \mathcal{A}_F^r$. Let be the Hochschild cochain dg-$\mathcal{F}^r$-algebra. There are canonical isomorphisms

\[(7.3.7) \quad \text{Hom}_{\mathcal{F}^r \text{-dgca}}(A^\vee, \text{Bar}(C)) \xrightarrow{\sim} \text{MC}_{\mathcal{F}^r}(C, A) \xrightarrow{\sim} \text{Def}_{nc}^{\mathcal{F}^r} (A).\]

Similarly, in analogy to Proposition 6.1.9 and its proof, we can find a tautological construction representing the deformation functor in terms of $E \wedge \rho$ the choice of idempotent made to construct $R_{nc}^{\mathcal{F}^r}$ and give $E \wedge \rho$ the structure of a $F^r$-algebra. We apply the Morita equivalence of categories explicitly given by $E \wedge \rho \text{-Mod} \xrightarrow{\sim} R_{nc}^{\mathcal{F}^r} \rho \text{-Mod}$

\[(7.3.8) \quad W \mapsto eW, \quad V \otimes F^r Y \leftrightarrow Y\]

generalizing the case $r = 1$ of (6.1.8).

**Proposition 7.3.9.** $\text{Def}_{nc}^{\mathcal{F}^r}$ is isomorphic to the $F^r$-inner automorphism quotient of the Hom$_{\mathcal{F}^r}$-functor on $A^\mathcal{F}^r$ of $R_{nc}^{\mathcal{F}^r}$. That is, there is a functorial isomorphism

\[
\text{Def}_{nc}^{\mathcal{F}^r}(A) \xrightarrow{\sim} \text{Hom}_{\mathcal{F}^r}(R_{nc}^{\mathcal{F}^r}, A)/\sim_{A,r}
\]

over $A \in \mathcal{A}_F^r$, where $\sim_{A,r}$ indicates the equivalence relation of inner $F^r$-algebra isomorphisms of $A$. This isomorphism is given by applying $(-) \otimes \text{End}_F(V)$ to a representative $\eta: R_{nc}^{\mathcal{F}^r} \rightarrow A$ of an element of $\text{Hom}_{\mathcal{F}^r}(R_{nc}^{\mathcal{F}^r}, A)/\sim_{A,r}$.

Finally, we prove that $\text{Def}_{nc}^{\mathcal{F}^r}$ is representable by $C$ when taken up to gauge equivalence and $F^r$-conjugation. The proof is exactly as in the 1-pointed version of Proposition 6.1.11. The one point of difference is that $F^r$ is no longer central as it was when $r = 1$, so the strict gauge action of conjugation by $1 + \text{End}_F(\rho) \otimes m_A$ must be followed by conjugation by $(F^r)^\times \cong \text{Aut}_E(\rho)$.

**Proposition 7.3.10.** We have the dg-$F^r$-algebra $C = C^\bullet(E, \text{End}_F(\rho))$. For $A \in \mathcal{A}_F^r$, there is the following natural bijection between $A$-valued Maurer-Cartan elements for $C$ and lifts of $\rho$ to $A$. That is, $\frac{\text{MC}(C, A)}{(F^r)^\times} \xrightarrow{\sim} \text{Def}_{nc}^{\mathcal{F}^r}(A)$.

### 7.4. $A_\infty$-algebras and deformations over multiple points

We now aim for an analogue of Theorem 7.3.6 expressing $\text{Def}_{nc}^{\mathcal{F}^r}$ in terms of cohomological data. We use the setup for the $r$-pointed Kadeishvili’s theorem and decomposition theorem from §7.2.5. After assembling a few more lemmas, we apply very similar arguments to the 1-pointed case to prove the main theorems.

From above, we have $E$, $\rho$, and $C$. Let

\[H := H^\bullet(C) = H^\bullet(E, \text{End}_F(\rho))\]

denote Hochschild cohomology of $\text{End}_F(\rho)$.

We choose a homotopy retract structure on $(H, 0)$ relative to $(C, d)_{C}$ as in Example 5.2.2 inducing
a minimal $A_\infty$-structure on $H = H^\bullet(G, \text{End}_F(\rho))$, denoted $(H, m) = (H, (m_n)_{n \geq 2})$,

extending its canonical graded algebra structure $m_2$. This comes along with

• a quasi-isomorphism $f : H \to C$ of $A_\infty$-algebras (Corollary [5.2.6]) and

• an isomorphism $\chi : C \to H \oplus K$ of $A_\infty$-algebras, where the projection to $H$ is a left inverse and right quasi-inverse to $f$ (Theorem [5.3.3]), and $(K, d_K)$ is a trivial $A_\infty$-algebra

We need the following $r$-pointed analogues of Lemmas 6.2.1 and 6.2.2.

**Lemma 7.4.1.** For a trivial $A_\infty$-algebra $(K, d_K)$, one has $\overline{MC}(K, -) = *$ and

\[
MC(K, A) \cong B^1 \otimes m_A
\]

for $A \in A^F_r$. For the particular trivial $A_\infty$-$F^r$-algebra $(K, d_K)$ produced from $C = C^\bullet(E, \text{End}_F(\rho))$, along with a homotopy retract on $C$ given in Example 7.2.3, we have

\[
MC(K, A) \cong \frac{\text{End}_F(\rho)}{\text{diag}(F^r)} \otimes m_A,
\]

where $\text{diag}(F^r)$ denotes the product of scalar matrices in $\text{End}_F(\rho)$.

**Proof.** Same as that of Lemma 6.2.1. \qed

**Lemma 7.4.2.** Assume that $H^i$ is finite-dimensional for all $i \in \mathbb{Z}$, so that $MC(H, -)$ is pro-represented on $A^F_r$ by the classical hull $R$ of $\text{Bar}^*(H) \in A^F_{\text{cts}}$ (see Example 5.5.7). For $A \in A^F_r$, this pro-representability maps strict equivalence classes in $MC(H, A)$ isomorphically onto strict $F^r$-inner automorphism classes in $\text{Hom}_F(R, A)$.

Just as $F^r$-inner automorphism of $A \in A^F_r$ means conjugation by $F^r \otimes A \cong \bigoplus_{i=1}^r A_{i,i}$, strict $F^r$-inner automorphism refers to conjugation by $1 + F^r \otimes m_A$.

**Proof.** The key point is that $H$ is itself augmented as an $A_\infty$-$F^r$-algebra, in the sense that $H^0 \cong F^r$, arising from the center of $C$. The proof then proceeds with the same calculations as in Lemma 6.2.2. We arrive at conjugation by $1 - \gamma$, where

\[
\gamma \in H^0 \otimes m_A \cong F^r \otimes m_A \cong \bigoplus_{i=1}^r (m_A)_{i,i}. \quad \Box
\]

Now we deduce an $r$-pointed analogue of Theorem 6.2.3.

**Theorem 7.4.3.** Let the data $E$, $\rho$, $C$, $H$ be as above. Choose an $r$-pointed homotopy retract between $H$ and $C$ as in Example 7.2.3, inducing $m$, $f$, and $\chi$.

For any $A \in A^F_r$, these data determine functorial isomorphisms

\[
\overline{MC}_{F^r}(H, A) / (F^r)^\times \xrightarrow{\sim} \overline{MC}_{F^r}(C, A) / (F^r)^\times \xrightarrow{\sim} \text{Def}_{\rho}^\text{nc}(A).
\]

As in the 1-pointed version, we write $\bar{f}_*$ to indicate the map on gauge equivalence classes induced from the map of Maurer-Cartan sets

\[
f_* : MC(H, -) \to MC(C, -) \cong \text{Def}_{\rho}^\text{nc}(\bigotimes (-))
\]

induced by $f : H \to C$. 


Proof. Analogously to the proof of Theorem 6.2.3, this follows straightforwardly from the $r$-pointed version of the decomposition Theorem 5.3.3 discussed in §7.2.5, the conclusions about the gauge action in Lemmas 7.4.1 and 7.4.2, and the rightmost isomorphism of the theorem statement from Proposition 7.3.10 □

Under the assumption that $H^i$ is has finite $\mathbb{F}$-dimension for all $i \in \mathbb{Z}$, we consider the augmented $\mathbb{F}r$-algebra

\begin{equation}
\hat{\mathcal{A}}r F_{nc} \rho \simeq (\Sigma H^1)^* \left( m^* ((\Sigma H^2)^*) \right),
\end{equation}

which is the classical hull of the augmented dg-$\mathbb{F}r$-algebra $\text{Bar}^*(H)$ that is dual to the co-dg-$\mathbb{F}r$-algebra produced by the bar construction (Fact 2.1.4). This directly generalizes the 1-pointed expression of (6.2.5).

Then, we prove an explicit relationship between $E^\wedge \rho$ and $R$.

**Corollary 7.4.5.** We assume the setting of Theorem 7.4.3, so that we have the data $E$, $\rho$, $C$, $H$, $m$, $f$, and $\chi$. Under the additional assumption that $H^i$ is finite-dimensional for all $i \geq 0$, these data

1) determine an isomorphism of $\mathbb{F}$-algebras

\[ \rho^u : E^\wedge \rho \longrightarrow \text{End}_{\mathbb{F}}(V) \otimes R \]

given by, for $x \in E$,

\[ x \mapsto \rho(x) + \sum_{i=1}^{\infty} (\varepsilon \mapsto (f_i(\varepsilon))(x)) \in \text{End}_{\mathbb{F}}(V) \otimes R \]

where $\varepsilon$ is a generic element of $(\Sigma H^1)^{\otimes i} = \Sigma H^i(E, \text{End}_{\mathbb{F}}(V))^{\otimes i}$.

2) Upon the additional choice of an idempotents used to give $E^\wedge \rho$ an $\mathbb{F}r$-algebra structure and define $R^nc r \rho$ (see (7.3.3)), $\rho^u$ induces an isomorphism in $\hat{\mathcal{A}}r F_{nc} \rho$,

\[ R^nc r \rho \simeq R. \]

The notation $(\varepsilon \mapsto (f_i(\varepsilon))(g))$ has mostly the same as in Corollary 6.2.6 with the sole exception that it is correct here to use $\otimes$: $f$ will send $\varepsilon_{i,j}$ to $\text{Hom}_{\mathbb{F}}(V_j, V_i)$, as we insisted that the homotopy retract respected $\mathbb{F}r$-structure.

**Remark 7.4.6.** As discussed in §4.5 Corollary 7.4.5 refines results of Segal [Seg08, Thm. 2.14] and Laudal [Lau02].

**Proof.** The proof proceeds just as the proof of Corollary 6.2.6. The formula for $\rho^u$ is identical, and respects $\mathbb{F}r$-structure because the retract structures and (therefore) the $A_{\infty}$-structures and homomorphisms do so.

We deduce the isomorphism (2), from which (1) follows. The choice of idempotents yields $e \rho^u : R^nc r \rho \rightarrow R$ exactly as in (6.2.8). Theorem 7.4.3 draws an isomorphism $\text{MC}(H, -) / (\mathbb{F}r)^x \simeq \text{Def}_{nc} \rho$. Proposition 7.3.9 shows that $R^nc r \rho$ represents $\text{Def}_{nc} \rho$ up to $\mathbb{F}r$-inner automorphism of the coefficients. By Proposition 7.3.7 there is an isomorphism of functors $\text{Def}_{nc} \rho \simeq \text{Def}_{nc} \rho$. Lemma 7.4.2 shows that the projection $\text{MC}(H, -) \rightarrow \text{MC}(H, -)$ amounts to $\mathbb{F}r$-inner automorphism classes in the coefficients in $\mathcal{A}_{\mathbb{F}}$. Recall that $R$ pro-represents $\text{MC}(H, -)$ on $\mathcal{A}_{\mathbb{F}}$ by its definition. Putting together these isomorphisms, we deduce that $R^nc r \rho$ and $R$ pro-represent functors on $\mathcal{A}_{\mathbb{F}}$ that are isomorphic via the map $f^*$ of Theorem 7.4.3.
(up to $\mathbb{F}^r$-inner automorphism). Because the formula for $\rho^u$ realizes the map $f_*$ discussed after Theorem 7.4.3, we see that $e\rho^u e$ is compatible with this isomorphism of functors, up to inner automorphism. Therefore $e\rho^u e$ is itself an isomorphism. \hfill $\square$

8. Massey products

The point of this section is to introduce Massey products, in preparation for the explanation of §9 of the relationship between lifts of representations and Massey products. Here, we focus on the relationship between Massey products and $A_\infty$-products; mainly, we follow \cite{LPWZ09}.

Remark 8.0.1. Massey products were first introduced in topology by Massey and Massey–Uehara \cite{Mas58, UM57}. For introductions relatively similar to our approach, see Kraines \cite{Kra66}, May \cite{May69}, and Dwyer \cite{Dwy75, Dwy75 §2}.

8.1. Massey products in dg-algebras. Let $C = (C^*, \partial, \sim)$ be some dg-$\mathbb{F}$-algebra, possibly non-unital as usual. Let $H = H^*(C)$ be its cohomology. A Massey product of degree $n$ is a multi-valued cohomology operation $H^{\otimes n} \to H$ of cohomological degree $2 - n$. They are not always defined: each value arises from a defining system. Presently, we will introduce these notions in detail.

The second Massey product $\langle , \rangle: H^{\otimes 2} \to H$ is unambiguously and unconditionally defined: is the reduction of $\sim$ module coboundaries, i.e. the cup product.

Further Massey products are defined as follows. We establish the notation $\bar{\sigma} = (-1)^{d+1}\sigma$ for $\sigma \in C^d$ for this general definition. Note that in our main case of interest where $d = 1$, we have $\bar{\sigma} = \sigma$.

Remark 8.1.1. There are at least two sign conventions used for Massey products. We follow May \cite{May69} and \cite{LPWZ09}, in contrast to \cite{Kra66} and \cite{LV12}.

Definition 8.1.2. Let $n \geq 3$. Let $I_n$ be the set of pairs of integers $(i, j)$ such that $1 \leq i \leq j \leq n$ and $(i, j) \neq (1, n)$.

Let $\sigma_i \in Z^{d_i} \subset C^{d_i}$ be cocycles for $1 \leq i \leq n$. For $(i, j) \in I \cup \{(1, n)\}$, let

$$d(i, j) = -(i - j) + \sum_{k=i}^{j} d_i.$$ 

We say that a set $S = \{\sigma(i, j) \in C^{d(i, j)} : (i, j) \in I\}$ is a defining system for the $n$-th Massey product $\langle \sigma_1, \ldots, \sigma_n \rangle$ if

1. $\sigma(i, i) = \sigma_i$ for all $i = 1, \ldots, n$, and

2. $\partial \sigma(i, j) = \sum_{k=i}^{j-1} \bar{\sigma}(i, k) \sim \sigma(k+1, j)$ for all $(i, j) \in I_n$ such that $i < j$.

When $S$ is a defining system for $\langle \sigma_1, \ldots, \sigma_n \rangle$, we note that

$$c(S) := \sum_{k=1}^{n-1} \bar{\sigma}(1, k) \sim \sigma(k+1, n)$$

is an element of $Z^{d(1, n)+1}$ and we let $\langle \sigma_1, \ldots, \sigma_n \rangle \in H^{d(1, n)+1}$ be the class of $c(S)$. We let

$$\langle \sigma_1, \ldots, \sigma_n \rangle = \{\langle \sigma_1, \ldots, \sigma_n \rangle_S \in H^{d(1, n)+1} \} \subset H^{d(1, n)+1}$$

where $S$ ranges over all defining systems. It may be empty.

Call $\langle \sigma_1, \ldots, \sigma_n \rangle$ defined if it is non-empty (i.e. there exists a defining system), and say that $\langle \sigma_1, \ldots, \sigma_n \rangle$ contains zero if $0 \in \langle \sigma_1, \ldots, \sigma_n \rangle$. 

It is known that the set $\langle \sigma_1, \ldots, \sigma_n \rangle$ only depends on the cohomology classes of $\sigma_1, \ldots, \sigma_n$ [Kra66, Thm. 3]. Also, note that $d(1, n) + 1 = 2 - n + \sum_{k=1}^n d_i$, confirming that the $n$-th Massey product has cohomological degree $2 - n$ (as a multi-valued map). Finally, note that $\langle \sigma_1, \ldots, \sigma_n \rangle$ is defined if and only if

- all lower-degree Massey products on proper sub-words of $\sigma_1 \sigma_2 \cdots \sigma_n$ are defined, and
- all of these contain zero.

In the sequel, we have $d_i = 1$ for all $i$; therefore all such Massey products are valued in $H^2$.

8.2. Massey powers in dg-algebras. Let $C$ continue to represent a dg-algebra. The following (non-standard) notion of Massey power will be useful for our representation-theoretic applications. Due to our attention to this application, we only discuss Massey powers of elements of $C^1$.

**Definition 8.2.1.** Let $\tau \in Z^1$, and let $\tau_1 := \tau, \tau_2, \ldots, \tau_{n-1} \in C^1$, where $n \geq 3$. We say that $T := \{\tau_1, \ldots, \tau_{n-1}\}$ is a defining system $S$ for the $n$-th Massey power $\langle \tau \rangle^n$ if the set

$$S = S(T) := \{\sigma(i, j) = \tau_j - i + 1 : 1 \leq i \leq j \leq n, (i, j) \neq (1, n)\}$$

is a defining system for the Massey product $\langle \tau, \ldots, \tau \rangle$ (with $\tau$ repeated $n$ times). A defining system (for the product) arising in this manner is called symmetric. If $T$ is a defining system for the Massey power $\langle \tau \rangle^n$, then we let $\langle \tau \rangle^n_T := \langle \tau, \ldots, \tau \rangle_S(T)$, and we let $c(T) := c(S)$. We let

$$\langle a \rangle^n = \{\langle a \rangle^n_T\} \subset H^2$$

where $T$ ranges over defining systems for the Massey powers.

We make the following important observations.

- $\langle \tau \rangle^n \subset \langle \tau, \ldots, \tau \rangle$, properly in general.
- $T = \{\tau_1, \ldots, \tau_{n-1}\} \subset C^1$ is a defining system for the Massey power $\langle \tau \rangle^n$ if and only if $\tau_1 = \tau$ and, for all $i = 1, \ldots, n-1$, we have

$$d \tau_i = \sum_{j=1}^{i-1} \tau_j - \tau_{i-j}. \tag{8.2.2}$$

- for such $T$, we have

$$c(T) = \sum_{j=1}^{n-1} \tau_j - \tau_{n-j}.$$ 

- The cohomology classes of $\langle a \rangle^n$ do not depend on the choice of $a$ within its cohomology class.

8.3. The relationship between Massey products and $A_\infty$-products. Recall from Definition 5.1.1 that an $A_\infty$-algebra structure $m$ on a graded $F$-vector space $H$ consists of maps $m_n : H^\otimes n \to H$ of homogeneous degree $2 - n$. Recall also from Corollary 5.2.6 that when $C = (C^\bullet, \partial, \sim)$ is a dg-$F$-algebra, then there are various compatible choices of $A_\infty$-algebra structure $m$ on its cohomology $H = H^\bullet(C)$. For example, homotopy retracts between $H$ and $C$ induce such an $m$, according to Example 5.2.8. This suggests a relationship between Massey products.
and $A_\infty$-products. Following [LPWZ09], we discuss the relationship between these two notions.

**Proposition 8.3.1.** Let $(C,d_C,m_C)$ be a dg-algebra with cohomology $H = H^*(C)$. We fix cohomology classes $a_i \in H^{d_i}$ for $1 \leq i \leq n$, where $n \geq 3$.

Choose in addition the data of a homotopy retract

$$h : (C,d_C) \xrightarrow{p} (H,0)$$

as in Example 5.2.2, specifying an $A_\infty$-algebra $(H,(m_n)_{n \geq 2})$ and a quasi-isomorphism $f : (C,d_C,m_C) \rightarrow (H,(m_n)_{n \geq 2})$ as in Example 5.2.7.

(a) Assume that for all $i$, $2 \leq i \leq n-1$ and all sub$i$-tuples $(a_j, \ldots, a_{j+i-1})$ of $(a_1, \ldots, a_n)$, it is the case that $m_i(a_j \otimes \cdots \otimes a_{j+i-1}) = 0$.

(b) Define

$$a(i,j) = f_{j-i+1}(a_1 \otimes \cdots \otimes a_j).$$

Then $(-1)^b m_n(a_1 \otimes \cdots \otimes a_n)$ is the element

$$\langle a_1, \ldots, a_n \rangle_D = \sum_{i=1}^{n-1} m_C(a(i,n-i) \otimes a(n-i,i))$$

of the Massey product $\langle a_1, \ldots, a_n \rangle$ arising from the defining system $D = \{a(i,j) : 1 \leq i \leq j \leq n, (i,j) \neq (1,n)\}$, where

$$b = 1 + d_{n-1} + d_{n-3} + \cdots$$

is a sum with final term $d_1$ or $d_2$.

We remark that condition (a) does depend on the choice of retract.

**Proof.** Using induction on $n$, we will show that the proposition follows directly from part (8) of Example 5.1.4. As loc. cit. notes, assumption (a) implies that the relation (5.1.5) holds when evaluated on $a_1 \otimes \cdots \otimes a_n$. Using the induction step, assumption (a) also implies that $d_C f_i(a_j \otimes \cdots \otimes a_{j+i-1})$ is equal to

$$-m_C(f_1 \otimes f_{i-1} - f_2 \otimes f_{i-2} + \cdots + (-1)^i f_{i-1} \otimes f_1)(a_j \otimes \cdots \otimes a_{j+i-1})$$

Using definition (b) for the $a(i,j)$, we see that (5.1.5) states that $\langle a_1, \ldots, a_n \rangle_D$ is a member of the cohomology class $m_n(a_1 \otimes \cdots \otimes a_n)$, as desired, up to some sign. From [LPWZ09] Thm. 3.1, this sign is given by $b$. □

**Example 8.3.4.** In particular, for our case of interest where $d_i = 1$ for all $i$, $b = (-1)^{(n+1)(n+2)/2}$. Moreover, when $d_i = 1$ for all $i$, this formula for $b$ extends to the case $n = 2$ (from $n \geq 3$ as in the statement of Proposition 8.3.1), where we have set the Massey product equal to the cup product.

The natural converse to Proposition 8.3.1 is not true, because there exist defining systems that cannot arise from a fixed set of $(f_i)$ as in (8.3.2). For example, consider a cup product $a \cup a = 0$, and a defining system $D$ for the triple Massey product $(a,a,a)_D = a(1,1) \sim a(2,3) + a(1,2) \sim a(3,3)$. If $a(1,2) \neq a(2,3)$, then (8.3.2) is not possible. Of course, sufficient conditions such that a Massey product arises as in (8.3.1b) can be made clear. For our purposes, it suffices to produce conditions guaranteeing that Massey powers arise from an $A_\infty$-structure arising from a homotopy retract.
Proposition 8.3.5. Let \((C, d_C, m_C)\) be a dg-algebra and choose \(\tau \in H^1(C)\). Let \(\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_{n-1}\}\) be a defining system for the Massey power \(\langle \tau \rangle^n\). The following claims are equivalent.

1. Then there exists a choice of retract of \((C, d_C)\) by \((H^*(C), 0)\) as in Examples 5.2.2 such that \(\tau_i = f_i(a^\otimes i)\).

2. There exists a section \(h^2 : B^2(C) \to C^1\) of \(d_C|_{C^1}\) such that \(h^2(\sum_{j=1}^{n-1} m_C(\tau_j \otimes \tau_{i-j})) = \tau_i\). In this case, \(m_n(a^\otimes n) = \langle \tau \rangle^n\).

Proof. Given \(h^2\) as in (2), one defines \(i_1 : H^1(C) \to C^1\) so that \(\tau_1 = i_1(\tau)\) and defines \(p^1 : C^1 \to H^1(C)\) so that it kills \(h^2(B^2(C))\) (which is possible because \(h^2(B^2(C))\) is linearly disjoint from \(\ker(d^1 : C^1 \to B^2)\)). Clearly these \(h^2, i_1, p^1\) can be extended to a retract \(h, i, p\) between \(H\) and \(C\). The converse is clear, in view of the Massey power defining system identities (8.2.2). 

\[\square\]

9. Massey products and associative deformations

The main point of this section is to illustrate that the Massey powers of \(\S 8.2\) in the Hochschild cohomology of \(\text{End}_\mathbb{F}(\rho)\) are intrinsically related to lifts of \(\rho\). Given the content of \(\S 6-7\) and the relationship between Massey powers and \(A_\infty\)-products given in \(\S 8.3\) this is no surprise. Thus the purpose of this discussion is to give an explicit and computable complement to the expression of associative deformation functors in terms of \(A_\infty\)-structures that are given in \(\S 6-7\). Indeed, we emphasize that a Massey power arises naturally when multiplying matrices in order to compute obstructions to lifts by hand.

For another treatment of the topic, which takes the approach of using Massey products to compute presentations for deformation spaces, see [Lau02] and the references therein.

9.1. Iterated extensions of representations. We will observe that Massey products arise when computing with certain extensions of representations.

Let \(\rho_i : E \to M_d(F)\), \(1 \leq i \leq n\), be a sequence of representations of \(E\). Let

\[\sigma_{i+1, i} \in Z^1(E, \text{Hom}_F(\rho_i, \rho_{i+1}))\]

for \(1 \leq i < n\)

be representatives of extension classes. Let \(d = \sum_{i=1}^n d_i\). Assume that there exists a representation \(\eta_n : E \to M_d(F)\) that realizes the \(\sigma_{i+1}\) below the diagonal, in the sense that there exist \(\sigma_{i,j}\) such that

\[
\eta_n = \begin{pmatrix}
\rho_1 & \sigma_{2,1} & \rho_2 \\
\sigma_{3,1} & \sigma_{3,2} & \rho_3 \\
\vdots & \ddots & \ddots \\
\sigma_{n,1} & \cdots & \sigma_{n,n-1} & \rho_n
\end{pmatrix}.
\]

We form a dg-\(F\)-algebra out of the Hochschild complex valued in the \(E\)-bimodule

\[\bigoplus_{1 \leq i < j \leq n} \text{Hom}_F(\rho_i, \rho_j)\]

with the natural compositions of homomorphisms making this \(E\)-bimodule a non-unital \(F\)-algebra. Now we use Massey products in this dg-algebra. One may readily
compute that $\eta_n$ is a homomorphism if and only if $\mathcal{S} = \{\sigma(i, j)\} = \{\sigma_{j,i}\}$ is a defining system for the $n$-th Massey product $\langle \sigma_{2,1}, \ldots, \sigma_{n,n-1}\rangle$ and

$$d\sigma_{n1} = c(\mathcal{S}).$$

In other words, we have an equivalence, as follows.

**Proposition 9.1.2.** There exists an $\eta_n$ realizing the $\sigma_{i+1,i}$ below the diagonal if and only if the Massey product $\langle \sigma_{2,1}, \ldots, \sigma_{n,n-1}\rangle$ is defined and contains zero.

This idea of such a connection between defining systems and extensions is due to May [May69].

This may be taken to be a condition on iterated extensions of representations: the condition that $\mathcal{S}$ is a defining system is equivalent to the existence of “overlapping” homomorphisms

$$\eta_{1,n-1} = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_{n-1} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n-1,1} & \sigma_{n-1,2} & \cdots & \sigma_{n-1,n-1} \end{pmatrix}, \quad \eta_{2,n} = \begin{pmatrix} \rho_2 & \rho_3 & \cdots & \rho_n \\ \sigma_{3,2} & \sigma_{3,3} & \cdots & \sigma_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_{n,n-1} \end{pmatrix}$$

Given this, the element $\langle \sigma_{2,1}, \ldots, \sigma_{n,n-1}\rangle \in c(\mathcal{S})$ of $\langle \sigma_{2,1}, \ldots, \sigma_{n,n-1}\rangle$ vanishes (in cohomology) if and only if there exists a common extension $\eta_n$ of $\eta_{1,n-1}$ and $\eta_{2,n}$. For $\eta_n$ exists if and only if there exists $\sigma_{n1} \in C^1(E, \text{Hom}_F(\rho_1, \rho_n))$ such that $d\sigma_{n1} = c(\mathcal{S})$.

### 9.2. Lifts of representations.

We will start with a representation $\rho : E \to M_d(\mathbb{F})$ as in §6.1.7. We will especially use the following coefficient algebras: for $n \geq 0$, write by $\mathbb{F}[\varepsilon_n] / (\varepsilon^{n+1}) \in C_d$.

An $n$-th order lift of $\rho$ is a lift $\rho_n$ of $\rho$ (as in Definition 6.1.1) to $\mathbb{F}[\varepsilon_n]$. We associate to $\rho_n$ an expression as a homomorphism to $M_{nd}(\mathbb{F})$, extending the standard basis $(a_i)_{i=1}^d$ of $\mathbb{F}^d$ to a $\mathbb{F}$-basis of $\mathbb{F}[\varepsilon_n]^d$ consisting of elements $\langle \varepsilon^j a_i : 1 \leq i \leq d, 0 \leq j \leq n \rangle$ with the ordering by $j$ and then by $i$. We arrive at the matrix realization

$$\rho_n = \begin{pmatrix} \rho & \sigma_1 & \sigma_2 & \cdots & \sigma_n \\ \rho & \sigma_1 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho & \sigma_1 & \vdots & \vdots & \vdots \\ \rho & \sigma_1 \end{pmatrix} : E \to M_{nd}(\mathbb{F}).$$

We will render this as

$$\rho_n = \rho + \sum_{i=1}^n \sigma_i \varepsilon^i : E \to M_{d}(\mathbb{F}[\varepsilon_n]),$$

where $\sigma_i$ is a function $\sigma_i : E \to M_d(\mathbb{F}) \cong \text{End}_\mathbb{F}(\rho)$.

Let $C = C^*(E, \text{End}_\mathbb{F}(\rho))$, so $\sigma_i \in C^1$. Because $\rho_n$ is a homomorphism, one readily observes that $\sigma_1$ lies in $Z^1$. More generally, the relations

$$d\sigma_i = \sum_{j=1}^{i-1} \sigma_j - \sigma_{i-j} \text{ for } 1 \leq i \leq n$$

are satisfied if and only if the corresponding expression for $\rho_n$ a homomorphism.
We may apply the connection between these conditions and Massey products from \([9.1]\). Next, we observe that these are Massey powers, using the symmetry visible by comparing \([9.2.1]\) to \([9.1.1]\). Namely, the set \(\mathcal{T} = \{\sigma_1, \ldots, \sigma_n\}\) satisfies \([9.2.2]\), and therefore it constitutes a defining system for the \((n+1)\)-st Massey power \(\langle \sigma_1 \rangle^{n+1}_{n} \). We will simply denote this Massey power defining system \(\mathcal{T}\) by \(\rho_n\) when it will not cause confusion. Thus the resulting Massey power is written \(\langle \sigma_1 \rangle^{n+1}_{n}\), the cohomology class of the cocycle \(c(\rho_n)\).

If we increment \(n\) to \(n+1\), the new relation of \([9.2.2]\) is

\[
\sigma_{n+1} = \sum_{j=1}^{n} s_j \sigma_{n-j+1} =: c(\rho_n),
\]

so there exists some \(\sigma_{n+1}\) satisfying \([9.2.2]\) if and only if \(c(\rho_n)\) is a 2-coboundary.

We summarize this discussion as follows.

**Proposition 9.2.3.** For \(n \geq 1\), let \(\rho_n\) be an \(n\)-th order lift of \(\rho\), defining cochains \(\sigma_i \in C^1(E, \text{End}_F(\rho))\) for \(1 \leq i \leq n\) as above. Then \(\sigma_1 \in Z^1(E, \text{End}_F(\rho))\) and the Massey power \(\langle \sigma_1 \rangle^{n+1}_n\) vanishes if and only if there exists an \(n+1\)-st order deformation \(\rho_{n+1} = \rho + \sum_{i=1}^{n+1} \sigma_i \varepsilon^i\) extending \(\rho_n\). In this case, we have an equality of 2-coboundaries \(d\sigma_{n+1} = c(\rho_n)\) and the set of possible \(\sigma_{n+1}\) is a \(Z^1(E, \text{End}_F(\rho))\)-torsor.

Varying over possible lifts extending a first order lift \(\rho_1\) of \(\rho\), we have the following immediate consequence, analogous to Proposition \([9.1.2]\).

**Corollary 9.2.4.** Let \(\rho_1 = \rho + \sigma_1 \varepsilon\) be a first order lift of \(\rho\). Then \(\rho_1\) extends to an \(n\)-th order lift if and only if the Massey power \(\langle \sigma_1 \rangle^n \subset H^2(E, \text{End}_A(\rho))\) is defined and contains zero.

9.3. **Expression of moduli spaces using Massey products.** In light of Proposition \([9.2.3]\) it is clear, in principle, that Massey products control \(\text{Def}^\text{nc}_n\). In this section, we explain how to compute universal lifts in terms of elements of Massey powers. We will only do this up to the point of illustrating the technique — and illustrating its limitations compared to the \(A_\infty\)-based expression — as we have already explained that \(A_\infty\)-structures give rise to Massey products and \(A_\infty\)-structures control \(\text{Def}^\text{nc}_n\). Indeed, see Remark \([9.3.4]\).

We observe that there is a universal first order lift \(\rho_1^n\) of \(\rho\). It is induced by the “universal 1-cocycle” defined by

\[
\sigma_1^n : E \to M_d(Z^1(E, \text{End}_F(\rho))^*), \quad \gamma \mapsto (\sigma_1 \mapsto \sigma_1(\gamma)_{i,j})_{i,j}
\]

for \(\gamma \in E\), \(\sigma_1 \in Z^1(E, \text{End}_F(\rho))\), and \((i,j)\) denoting the matrix coordinate. There is a left and a right \(E\)-action on \(M_d(Z^1(E, \text{End}_F(\rho))^*) \cong M_d(\mathbb{F}) \otimes Z^1(E, \text{End}_F(\rho))^*\) given by the usual left and right actions of \(E\) on \(M_d(\mathbb{F})\) via \(\rho\) and the multiplication map of \(M_d(\mathbb{F})\).

Letting \(\mathbb{F}[M] \) denote the square-zero \(\mathbb{F}\)-algebra extension of \(\mathbb{F}\) by an \(\mathbb{F}\)-vector space \(M\), we have

\[
\rho_1^n = \rho + \varepsilon \sigma_1^n : E \to M_d(\mathbb{F}[Z^1(E, \text{End}_F(\rho))^*]).
\]

This lift is universal in the sense that for any other first-order lift \(\rho_A\), there is a unique \(\mathbb{F}\)-linear map \(E^1(E, \text{End}_F(\rho))^* \to M_A\) such that \(\rho_A \otimes \mathbb{F}[Z^1(E, \text{End}_F(\rho))^*] A = \rho_A\). This follows from the fact that \(\rho_A - \rho \otimes \mathbb{F}[Z^1(E, \text{End}_F(\rho))^*] A = 0\) in \(Z^1(E, \text{End}_F(\rho) \otimes \mathbb{F} M_A)\).
For the remainder of this section we produce a universal lift of any order, applying the computations of the previous section.

We establish notation for the sake of concision: write $T$ for $Z^1(E, \text{End}_\mathbb{F}(\rho))$ and let $\mathbb{F}[T_n] := \bigoplus_{i=0}^n T^\otimes i$ be the free associative $\mathbb{F}$-algebra on $T$ truncated at degree $n$, in analogy with $\mathbb{F}[\varepsilon_n]$. Inductively, we construct Massey powers of $\sigma^u \in Z^1(E, \text{End}_\mathbb{F}(\rho)) \otimes T$. The base case is the cup product, which is the cohomology class of the unambiguously defined 2-cocycle
\[
\sigma^u_1 \cup \sigma^u_1 \in H^2(E, \text{End}_\mathbb{F}(\rho)) \otimes T^\otimes 2.
\]
Let $I_2 \subset T^\otimes 2$ be the minimal submodule such that $\sigma^u_1 \cup \sigma^u_1$ vanishes modulo $I_2$. Then we can solve (9.2.2) modulo $I_2$, i.e. there exists $\sigma^u_2 \in C^1(E, M_d(T^\otimes 2/I_2))$ such that
\[
d\sigma^u_2 = \sigma^u_1 \cup \sigma^u_1 \pmod{I_2}
\]
and the set of possible choices for $\sigma^u_2$ is a torsor under $Z^1(E, \text{End}_\mathbb{F}(\rho)) \otimes \mathbb{F}T^\otimes 2/I_2$. We get a second order lift
\[
\rho^u_2 = \rho + \sigma^u_1 + \sigma^u_2 : E \to M_d(\mathbb{F}[T_2]/I_2).
\]
In fact, untangling dualities, we see that $\rho^u_2$ is a torsor under $Z^1(E, \text{End}_\mathbb{F}(\rho)) \otimes \mathbb{F}T^\otimes 2/I_2$.

We establish notation for the sake of concision: write $\mathbb{F}[T_n]$ for $Z^1(E, \text{End}_\mathbb{F}(\rho)) \otimes \mathbb{F}T^\otimes n$, and let $I_n$ be the minimal submodule of $T^\otimes n$ such that $\sigma^u_1 \cup \sigma^u_1 \cup \ldots \cup \sigma^u_1$ vanishes modulo $I_n$. The inductive step from order $n$ to order $n+1$ is to start with an $n$-th order lift $\rho^u_n = \rho + \sum_{i=1}^n \sigma^u_i$ of $\rho$ with coefficients in
\[
\frac{\mathbb{F}[T_n]}{(I_2, I_3, \ldots, I_n)}
\]
and calculate Massey power $\langle \sigma^u_1 \rangle_{\rho^u_n}$ valued in $H^2(E, \text{End}_\mathbb{F}(\rho)) \otimes \mathbb{F}T^\otimes n+1/I_{n+1}$, where $I_{n+1}$ is the ideal of $\mathbb{F}[T_{n+1}]$ generated by the image of $(I_2, I_3, \ldots, I_n) \subset \mathbb{F}[T_n]$ under the (non-multiplicative) natural map $\mathbb{F}[T_n] \to \mathbb{F}[T_{n+1}]$. Then define $I_{n+1}$ to be the minimal submodule of $T^\otimes n+1$ containing the degree $n+1$ projection of $I_{n+1}$ and such that $\langle \sigma^u_1 \rangle_{\rho^u_n}$ vanishes modulo $I_{n+1}$. As in the case $n = 1$, we now have $\sigma^u_{n+1}$ valued in $T^\otimes n/I_{n+1}$ and $\rho^u_{n+1}$ valued in $\mathbb{F}[T_{n+1}]/(I_2, I_3, \ldots, I_{n+1})$.

Because $I_n$ is concentrated in degree $n$, we have limits
\[
R^{\square}_\rho := \varprojlim_n \frac{\mathbb{F}[T_n]}{(I_2, I_3, \ldots, I_n)}, \quad \rho^{\square} := \varprojlim_n \rho^u_n : E \to M_d(R^{\square}_\rho)
\]
where $(R^{\square}_\rho, \mathfrak{m}^{\square}_\rho)$ is a complete local $\mathbb{F}$-algebra quotient of $\mathbb{T}_\rho$.

We summarize the construction above and state its universal property.

**Theorem 9.3.3.** For any local Artinian $\mathbb{F}$-algebra with residue field $\mathbb{F}$ and lift $\rho_A$ of $\rho$ valued in $A$, there exists a unique local $\mathbb{F}$-algebra homomorphism $R^{\square}_\rho \to A$ such that $\rho_A = \rho^{\square} \otimes \hat{R}^{\square}_\rho$ $A$. That is, $\text{Def}^{\square}_\rho = \text{Spf} \hat{R}^{\square}_\rho$ and $R^{\square}_\rho$ pro-represents $\text{Def}^{\square}_\rho$.

Moreover, we have a canonical isomorphism $(\mathfrak{m}^{\square}_\rho/(\mathfrak{m}^{\square}_\rho)^2)^* \cong Z^1(\text{End}_\mathbb{F}(\rho))$.

We omit the proof, since it amounts to the same argument as the proof of Corollary 6.2.6, but is more complicated due to the inductive construction of $R^{\square}_\rho$. 
Remark 9.3.4. Indeed, the main point of the illustration is to allow comparison with the simpler construction of Corollary [6.2.6]. The comparison rests on Proposition [8.3.1]: the choice of a homotopy retract “chooses all Massey products in advance,” because it chooses all $A_\infty$-products and also induces the data of Massey defining systems of degree $n + 1$, when the appropriate degree $n$ product vanishes. The inductive expression reflects that we choose an arbitrary defining system for a Massey power, at each step.

Remark 9.3.5. One advantage of Massey powers over $A_\infty$-products is that there is no problem with replacing the base coefficient ring $\mathbb{F}$ with a general commutative ring $S$. Indeed, all of the calculations of ideals $I_n$ make sense in this case, and the ideals $I_n$ may have non-trivial $S$-part. For example, when $S = \mathbb{Z}/p^2$ for some prime $p$, it is possible for a non-trivial first-order lift to exist modulo $p^2$ – i.e. over $S[\epsilon]/(p\epsilon^3, \epsilon^4)$ – an no extension to a third order lift to exist at all. If this particular first-order lift is unique modulo $p$, the universal deformation ring in this case would be $R$.

Example 9.3.6. Massey powers have been used in the generality of Remark 9.3.5 to calculate an invariant that controls congruences of modular forms, answering a question of Mazur. This is a main theorem of the author’s joint with Wake [WWE17c]; see §13.4.

Remark 9.3.7. One obstruction to applying the technology of $A_\infty$-algebras, as in §5 over a general commutative base ring $S$ in place of $\mathbb{F}$ is that homotopy retracts as in Example [5.2.2] may not exist. A replacement for Kadeishvili’s theorem (Corollary 5.2.6) is needed.

Part 3. Moduli of Galois representations and pseudorepresentations

In this part, we apply the results of Part 2 to cases of interest in number theory. First we set up the theory for representations of a profinite group. Continuity of representations is taken to be implicit, and we now discuss only commutative coefficients. Then we adapt this theory for cases of interest in number theory: representations of a profinite group satisfying some additional condition.

10. Moduli of representations of a profinite group

We recall moduli spaces of representations of a profinite group. These moduli spaces were set up in the author’s previous work [WE18], which we now recall. In contrast with loc. cit., we work in constant positive characteristic. Thus the base field is a finite field. We write $p$ for the characteristic.

10.1. Connected components biject with residual semi-simplification. The main result of [WE18, §3] is that the moduli of (integral) $p$-adic representations of a profinite group is the disjoint union of connected components parameterized precisely by the residual semi-simplification. We set up a precise meaning of the term “residual semi-simplification.”

Definition 10.1.1. Let $G$ be a profinite group and let $p$ be a prime. A residual semi-simplification is an equivalence class of semi-simple representations of $G$
valued in a finite field $\mathbb{F}$ of characteristic $p$ such that each simple summand is absolutely irreducible. The equivalence relation is isomorphism of representations, or, equivalently, isomorphism after change of coefficients via $\otimes_{\mathbb{F}}$. A residual semi-simplification is called *multiplicity-free* if there are no non-trivial isomorphisms among the simple summands.

**Remark 10.1.2.** Residual semi-simplifications are in bijection with residual pseudo-representations in [WE18, Def. 3.4].

Fix a representative $\rho : G \to \text{GL}_d(\mathbb{F})$ of a residual semi-simplification; we take $\mathbb{F}$ to be the smallest possible base field, writing it as $\mathbb{F}_\rho$ when needed for clarity. We set up the equal-characteristic moduli of deformations of all residual representations with residual semi-simplification $\rho$. Without loss of generality, we take $\rho$ to be in block diagonal form in $\text{GL}_d$, with diagonal summands

$$\rho \cong \bigoplus_{i=1}^r \rho_i.$$ 

Here $\rho_i : G \to \text{GL}_{d_i}(\mathbb{F})$ are the absolutely irreducible factors of $\rho$. Given this data, we write $\text{GL}(\rho)$ for the corresponding Levi sub-$\mathbb{F}$-algebraic group

$$\text{GL}(\rho) := \prod_{i=1}^r \text{GL}_{d_i} \hookrightarrow \text{GL}_d.$$ 

Write $\text{PGL}(\rho)$ for the quotient group of $\text{GL}(\rho)$ by the center of $\text{GL}_d$.

The natural equal-characteristic category of coefficient rings are *topologically finitely generated* $\mathbb{F}_p$-algebras, i.e. topological quotients of algebras of the form $\mathbb{F}_p[x_1, \ldots, x_n][y_1, \ldots, y_m]$, where the topology is $(x_1, \ldots, x_n)$-adic. We denote this category by $\text{Aff}_{\mathbb{F}_p}$. Equivalently, $\text{Aff}_{\mathbb{F}_p}$ is the category of finite type affine $\text{Spf} \mathbb{F}_p$-formal schemes. It is natural to replace $\mathbb{F}_p$ by $\mathbb{F}$ when we impose the condition that a representation has residual semi-simplification $\rho$, as follows.

**Definition 10.1.3.** Let $\text{Rep}_d^\square$ denote the functor on $A \in \text{Aff}_{\mathbb{F}_p}$ valued in sets, given by

$$\text{Rep}_d^\square(A) = \{\rho_A : G \to \text{GL}_d(A)\}.$$ 

Likewise, we have the quotient groupoid by the adjoint action of $\text{PGL}_d$,

$$\text{Rep}_d := [\text{Rep}_d^\square / \text{PGL}_d].$$ 

Let $\text{Rep}_d^\square$ denote the subfunctor of $\text{Rep}_d^\square \times_{\mathbb{F}_p} \mathbb{F}_\rho$ given by

$$\text{Rep}_d^\square(A) = \{\rho_A : G \to \text{GL}_d(A) \mid \text{for all } f : A \to \overline{\mathbb{F}}, (\rho_A \otimes_{\mathbb{F}, f} \overline{\mathbb{F}})^{ss} \simeq \rho \otimes_{\mathbb{F}} \overline{\mathbb{F}}\}$$

for $A \in \text{Aff}_{\mathbb{F}_p}$. Here $\simeq$ indicates being in the same orbit under the adjoint action of $\text{PGL}_d(A)$.

Finally, let $\text{Rep}_\rho$ denote the quotient formal stack

$$\text{Rep}_\rho := [\text{Rep}_d^\square / \text{PGL}_d],$$

A main result of [WE18] is that the mixed-characteristic versions of such spaces are representable in the category of $\text{Spf} \mathbb{Z}_p$-formal schemes. We state this result specialized to equal-characteristic.
Theorem 10.1.4 ([WE18 §3.1]). \( \text{Rep}_p \) is representable by a topologically finite type affine \( \text{Spf} \mathcal{F} \)-formal scheme \( \text{Spf} S_\rho \). We have

\[
\text{Rep}_d \times_{S_\rho} \mathcal{F}_p = \coprod_{\rho} \text{Rep}_\rho \times_{S_\rho} \mathcal{F}_p,
\]

where \( \rho \) varies over all residual semi-simplifications of dimension \( d \) and \( \mathcal{F}_\rho \) denotes the coefficient field of \( \rho \).

In additional to representability, the main upshot is that in order to understand the entire moduli space \( \text{Rep}_d \), we may study it one residual semi-simplification \( \rho \) at a time. Similarly, this theorem implies algebraicity and decomposition into connected components parameterized by \( \rho \) for the stack quotients \( \text{Rep}_d \).

10.2. Theory of pseudorepresentations, Cayley-Hamilton algebras, and generalized matrix algebras. We review the theory of pseudorepresentations due to Chenevier [Che14]. Because the review of [WE18 §2] is precisely what we need, we refer the reader there. Here, we recall only notation and selected parts of definitions.

- \( D : E \to A \) denotes a pseudorepresentation. Using this notation implies that \( E \) is an associative unital \( A \)-algebra, where \( A \) is a commutative ring. This \( D \) has a dimension \( d \in \mathbb{Z}_{\geq 1} \), and is a functor from commutative \( A \)-algebras \( B \) to functions \( D_B : E \otimes_A B \to B \) that are homogeneous of degree \( d \) in \( B \).
- When \( G \) is a group, \( D : G \to A \) is notation for a pseudorepresentation \( D : A[G] \to A \).
- For any commutative \( A \)-algebra \( B \), and \( x \in E \otimes_A B \), there is a characteristic polynomial \( \chi_D(x,t) \in B[t] \).
- Given \( D : E \to A \), there is a notion of a kernel two-sided ideal and Cayley-Hamilton two-sided ideal of \( E \),

\[
E \supset \ker(D) \supset \text{CH}(D).
\]

There exist canonical factorizations of \( D \) through \( E/\ker(D) \) and \( E/\text{CH}(D) \).

- A pseudorepresentation is called Cayley-Hamilton when \( \text{CH}(D) = 0 \). Equivalently, for all commutative \( A \)-algebras \( B \) and all \( x \in E \otimes_A B \), \( \chi_D(x,x) = 0 \). That is, \( x \) satisfies its own characteristic polynomial \( \chi_D(x,t) \in B[t] \). Collectively, such data \( (E,A,D : E \to A) \) is called a Cayley-Hamilton \( A \)-algebra.
- Given a representation \( \eta : G \to M_d(A) \), there is an induced \( d \)-dimensional pseudorepresentation, denoted \( \psi(\eta) : G \to A \), given by composing \( \eta \) with the determinant pseudorepresentation \( \text{det} : M_d(A) \to A \).
- Similarly, if \( E \) is equipped with a pseudorepresentation \( D : E \to A \) and \( \eta \) is a homomorphism \( G \to E^\times \), then there is an induced pseudorepresentation \( D \circ \eta : G \to A \). We especially study the case where \( (E,A,D) \) is a Cayley-Hamilton representation. Then we call the data

\[
(\eta : G \to E^\times, E,A,D : E \to A)
\]

a Cayley-Hamilton representation of \( G \), and we call \( \psi(\eta) := D \circ \eta \) its induced pseudorepresentation .

- A generalized matrix algebra over \( A \) or \( A \)-GMA is an associative \( A \)-algebra \( E \) equipped with
  - a complete orthogonal set of idempotents \( (e_i)_{i=1}^{r} \subset E \),
  - \( A \)-algebra isomorphisms \( e_i E e_i \to M_{d_i}(A) \)
that satisfy an extra condition. This notion, due in this form to Bellaïche–
Chenevier [BC09 §1], was shown to admit a natural Cayley-Hamilton pseudo-
representation \( D_{\text{GMA}} : E \to A \) in [WE18 Prop. 2.23].

- We call a Cayley-Hamilton representation \((\rho, A, E, D)\) a **GMA representation** over \( A \) when \( E \) is also equipped with GMA data such that \( D = D_{\text{GMA}} \).

The application of the tools above to the moduli of profinite groups is the main
content of [WE18 §3]. We require a few more definitions and results about this
situation, which we recall directly from [WE18 §3]. The results about deformation
theory of pseudorepresentations are due to Chenevier [Che14]. However, here we
work in constant characteristic \( p \).

We start with a fixed residual semi-simplification \( \rho : G \to \GL_d(\mathbb{F}) \) and its induced
\( d \)-dimensional pseudorepresentation \( D = \psi(\rho) : G \to \mathbb{F} \).

- There is a deformation functor sending \( A \in \mathcal{C}_\mathbb{F} \) to pseudorepresentations \( D_A : G \to A \) such that the reduction modulo \( m_A \)
\[
G \xrightarrow{D_A} A \twoheadrightarrow \mathbb{F}
\]
is equal to \( D \). Such a \( D_A \) is called a **pseudodeformation of \( D \) to \( A \)**. This gives rise to a deformation functor on \( \mathcal{C}_A \) for \( D \).

- There is a universal pseudodeformation ring \( R_D \) representing the deformation
functor for \( D \). It also represents the extension of this deformation problem
from \( \mathcal{C}_\mathbb{F} \) to \( \text{Aff}_\mathbb{F} \). Thus \( R_D \) supports the universal deformation
\[
D^u : G \to R_D
\]
of \( D \). It is a complete local \( \mathbb{F} \)-algebra with residue field \( \mathbb{F} \). When \( G \) satisfies
the \( \Phi_p \) finiteness condition of [Maz89 §1], \( R_D \) is Noetherian.

- We say that a Cayley-Hamilton representation
\[
(\eta : G \to E^\times, A, E, D_E : E \to A)
\]
where \( A \in \mathcal{C}_\mathbb{F} \) has residual pseudorepresentation \( D \) when its induced pseudorepresentation
\[
\psi(\eta) = D_E \circ \eta : E \to A
\]
is a deformation of \( D \). For short, we say that \( \rho \) is over \( D \). These notions have
a sensible extension of coefficient algebras from \( A \in \mathcal{C}_\mathbb{F} \) to \( A \in \text{Aff}_\mathbb{F} \).

- There is a universal Cayley-Hamilton representation of \( G \) over \( D \), produced as follows. We let \( E_D \) be the Cayley-Hamilton quotient of the universal pseudo-
deformation of \( D \), that is,
\[
E_D := \frac{R_D[G]}{\text{CH}(D^u)}.
\]
The theory of Cayley-Hamilton algebras recalled above implies that \( D^u \) factors through \( E_D \) as a Cayley-Hamilton pseudorepresentation; we denote the factorization by \( D_{E_D} \). Thus we have a Cayley-Hamilton pseudorepresentation
\[
(\rho^u : G \to E_D^\times, R_D, E_D, D_{E_D} : E \to R_D)
\]
over \( D \). Its induced pseudorepresentation \( \psi(\rho^u) : G \to R_D \) is equal to the
universal pseudodeformation \( D^u : G \to R_D \) of \( D \).

- When \( G \) satisfies the \( \Phi_p \) finiteness property, \( E_D \) is finitely generated as a \( R_D \-
module. Therefore it is a Noetherian ring.

10.3. **Application to moduli spaces of representations.** In order to apply the theory of Cayley-Hamilton representations to the moduli spaces of representations $\text{Rep}_\rho$, $\text{Rep}_\mu$, we make the following observations and additional definitions, which come from [WE18, §3].

- When $D = \psi(\rho) : G \to \mathbb{F}$, there is a natural functor $\psi : \text{Rep}_\rho \to \text{Spf} R_D$ sending a representation $\eta : G \to M_d(A)$ with residual semi-simplification $\rho$ to its induced pseudorepresentation $\psi(\eta) = \det \circ \eta : G \to A$, which is a pseudodeformation of $D$. And $\psi$ factors through $\psi : \text{Rep}_d \to \text{Spf} R_D$.

- Given a $d$-dimensional Cayley-Hamilton algebra $(E, A, D : E \to A)$, there exists an affine $A$-scheme $\text{Rep}_{E,D}$ of representations of $E$ that are compatible with $D$. It is a functor on commutative $A$-algebras $B$ sending $B \mapsto \{ \eta : E \to M_d(B) \mid \psi(\eta) := \det \circ \eta : E \to B$ equals $D \otimes_A B : E \to B \}$.

Likewise, there is an explicit moduli groupoid of Azumaya algebra-valued representations, which is represented by the stack quotient $[\text{Rep}_{E,D}/\text{PGL}_d]$.

- Given a $d$-dimensional GMA $(E, A, D_{\text{GMA}} : E \to A)$ with idempotents $e_i$ each of dimension $d_i$, there exists a closed sub-$A$-scheme

$$\text{Rep}_{E,D} \subset \text{Rep}_{E,D}$$

of adapted representations. The notion and moduli of adapted representations were first studied in [BC09, §1.3]. These are matrix algebra-valued representations that fix the data of idempotents, where we choose a diagonal data of idempotents in the matrix algebra.

- Let $Z(e_i)$ be the split torus in $\text{GL}_d$ which centralizes the block diagonal sub-algebra

$$\bigoplus_{i=1}^r e_i E e_i \cong \bigoplus_{i=1}^r M_{d_i}(A) \hookrightarrow M_d(A).$$

This torus has a natural adjoint action on $\text{Rep}_{E,D}^{\text{GMA}}$, and its stack quotient admits an isomorphism

$$[\text{Rep}_{E,D}^{\text{GMA}}/Z(e_i)] \cong \text{Rep}_{E,D}.$$  

- Let $e_i^{11}$ denote the idempotent of $e_i E e_i \cong M_{d_i}(A)$ cutting out the $(1, 1)$-coordinate of $M_d(A)$. Let $e^{11} = \sum_{i=1}^r e_i^{11}$. We then get a Morita-equivalent algebra

(10.3.1) $$e^{11} E e^{11}$$

that naturally admits the structure of a GMA: the idempotents are $e_i^{11}$ and $d_i = 1$ for all $1 \leq i \leq r$. We write $A_{i,j}$ for

$$A_{i,j} := e_i^{11} E e_{i}^{11} = e_j^{11} (e^{11} E e^{11}) e_i^{11},$$

- According to [BC09, Prop. 1.3.9], there is an expression for the $A$-algebra $S_{E,D}^{\text{GMA}}$ representing the affine $A$-scheme $\text{Rep}_{E,D}^{\text{GMA}}$ in terms of the multiplication map on $e^{11} E e^{11}$ decomposed into its idempotent-based coordinates as

$$\varphi_{i,j,k} : A_{i,j} \otimes_A A_{j,k} \to A_{i,k}.$$
The expression for $S_{E,D}^{GMA}$ is

\[(10.3.2) \quad S_{E,D}^{GMA} \sim \text{Sym}_A^* \left( \bigoplus_{1 \leq i \neq j \leq r} A_{i,j} \right) / (x \otimes y - \varphi(x \otimes y)), \]

where the denominator stands for $x \in A_{i,j}$, $y \in A_{j,k}$, and $\varphi = \varphi_{i,j,k}$, as $(i,j,k)$ and $(x,y)$ vary.

We summarize the results about these objects given in [WE18].

**Theorem 10.3.3** ([WE18 §3]). Let $\rho$ be a residual semi-simplification with $D = \psi(\rho)$, and assume that $G$ satisfies the $\Phi_p$ finiteness condition. There are natural isomorphisms of $\text{Spf} \mathbb{F}$-spaces

\[\text{Rep}_\rho \cong \text{Rep}_{E,D}^{E,D}, \quad \text{Rep}_\rho \cong \text{Rep}_{E,D}^{E,D},\]

each of which admits a finite type module over $\text{Spec} R_D$. That is, there is an isomorphism of the moduli of

- representations of $G$ with residual semi-simplification $\rho$ and
- representations of $E_D$ that are compatible with the pseudorepresentation $D^n : E_D \to R_D$,

which is an isomorphism of $\text{Spf} R_D$-formal spaces.

Furthermore, assuming that $\rho$ is multiplicity-free,

1. $E_D$ admits the structure $(e_i)_{i=1}^r$ of an $R_D$-GMA such that $D^n = D_{GMA}$ and

\[\text{Rep}_\rho \cong [\text{Rep}_{E,D}^{E,D} / Z(e_i)]\]

2. If we let $S_{\rho}^{GMA}$ be the commutative $A$-algebra representing the affine scheme

\[\text{Rep}_{E,D}^{E,D}, \quad \text{then the structure morphism } \psi_{GMA} : \text{Rep}_{E,D}^{E,D} \to \text{Spf} R_D \text{ induces an isomorphism } R_D \sim (S_{E,D}^{GMA})^{\psi_{GMA}}.\]

**Remark 10.3.4.** As discussed in [WE18 §3], the result (2) means that $\text{Spec} R_D$ is a GIT quotient for the stack $\text{Rep}_\rho$. Even when $\rho$ is not multiplicity-free, so that (2) is not known to hold, there is a map $R_D \to (S_{\rho}^{\square})^{\text{PGL}_d}$ (where $\text{Rep}_\rho \cong \text{Spf} S_{\rho}^{\square}$) that is very close to being an isomorphism.

**Proof.** The initial statements come from [WE18 Thm. 3.7]. The GMA structure claimed in (1) comes from [Che14 Thm. 2.22(ii)]. The rest of (1) is proved in [WE18 Thm. 2.27], and (2) is [WE18 Thm. 3.8(4)].

**11. Presentations in terms of $A_\infty$-structure on group cohomology**

We express $\text{Rep}_\rho^{\square}$ as a moduli space of representations of an algebra, so that we may apply the results of Part 2.

11.1. **From Hochschild cohomology to group cohomology.** Given a left $G$-module $V$, we write $C^* (G,V)$ for the cochain complex of inhomogeneous group cochains; see [2.2] or, for a full introduction, see e.g. [BroS2]. Because we are working over a field, group cohomology realizes the Ext-functors in the category of $\mathbb{F}[G]$-modules. There is also a direct compatibility between group cohomology and Hochschild cohomology of left modules for the group algebra.

(1) There is an isomorphism from the Hochschild complex to the group cochain complex

$$\theta^n : C^n(\mathbb{F}[G], \text{Hom}_\mathbb{F}(W, V)) \xrightarrow{\sim} C^n(G, \text{Hom}_\mathbb{F}(W, V))$$

using the natural embedding $G \times^n \hookrightarrow \mathbb{F}[G] \otimes^n$.

(2) There are canonical isomorphisms of graded $\mathbb{F}$-vector spaces

$$H^\bullet(\mathbb{F}[G], \text{Hom}_\mathbb{F}(W, V)) \xrightarrow{\sim} H^\bullet(G, \text{Hom}_\mathbb{F}(W, V)) \xrightarrow{\sim} \text{Ext}^\bullet_{\mathbb{F}[G]}(V, W).$$

Proof. Because $\mathbb{F}[G] \otimes^n \cong \mathbb{F}[G \times^n]$, $\theta$ is an isomorphism of graded vector spaces. The differentials are compatible under $\theta$ because the map from $\mathbb{F}[G]$-bimodule actions $\ast = (\ast_{\text{left}}, \ast_{\text{right}})$ to left $G$-module actions

$$g \cdot m := g \ast_{\text{left}} m \ast_{\text{right}} g^{-1}$$

sends the given $\mathbb{F}[G]$-bimodule action on $\text{Hom}_\mathbb{F}(W, V)$ to its standard left $G$-module action. From there, one observes that the formula for the Hochschild differential (Definition 6.1.2) is the same as the formula for the differential on inhomogeneous group cochains.

The leftmost isomorphism of (2) is then clear. The right isomorphism relies on $C^n(G, \mathbb{F})$ being a projective resolution for the $G$-module $\mathbb{F}$, and $\mathbb{F}$ having trivial homological dimension. □

Remark 11.1.2. The arguments are valid for discrete modules with a continuous action of $G$, when $G$ is a profinite group, using the fact that $\mathbb{F}$ is finite. The key fact is that $C^n(G, \mathbb{F})$ remains a resolution; see e.g. [RZ10, Prop. 6.2.2]. Correspondingly, the ambient categories are continuous finite discrete $G$-modules, resp. continuous finite discrete $\mathbb{F}[G]$-modules.

11.2. Presentation of the completed group algebra. Note that each absolutely irreducible factor $\rho_i$ of $\rho$ cuts out a maximal ideal of the completed group algebra $\mathbb{F}[\hat{G}]$ (see e.g. [RZ10, §5.3] for the definition). We now let

$$E := \mathbb{F}[\hat{G}]$$

and apply the theory of Part 2 as follows.

- We consider only continuous representations of $E$, but other than in this paragraph we leave this implicit without stating it explicitly.
- Likewise, we let $\mathcal{C} = C^\bullet(E, \text{End}_\mathbb{F}(\rho))$ denote the continuous Hochschild cochain complex

$$C^i(E, \text{End}_\mathbb{F}(\rho)) := \text{Hom}_{\text{cts}}(E^{\otimes i}, \text{End}_\mathbb{F}(\rho)).$$

It is a straightforward exercise to check that the differential and multiplication in $C$ preserves continuity.

After setting up these two instances of continuity, there are no additional instances where we must impose it. For consider that the lift $\rho \otimes \xi$ of $\rho$ to $A \in A_\mathbb{F}$ associated to a Maurer-Cartan element $\xi \in \text{MC}(C, A) \subset C^1 \otimes m_A$ is obviously continuous. As all other representations are ultimately produced out of elements of $C^1$ along with formulas within $C$ that preserve continuity, namely, those of Example 5.2.8, we will implicitly always work in the continuous case in the sequel.
Theorem 11.2.1. Assume that $G$ satisfies the $\Phi_p$ finiteness condition. Choose an $r$-pointed homotopy retract between $C^\bullet(E, \text{End}_F(\rho))$ and its cohomology $H = H^\bullet(C)$, as in Example 7.2.3. Choose also idempotents as in (7.3.3), including $e$. These choices induce an $A_\infty$-algebra structure $m$ on $H$ and a complete $F$-algebra isomorphisms

$$\rho^\nu : F[G]^\wedge_{\rho} \xrightarrow{\sim} \text{End}_F(V) \otimes \frac{\hat{T}_F(\Sigma H^1)^*}{(m^*((\Sigma H^2)^*))},$$

$$e\rho^\nu e : R_{\rho}^{nc} := eF[G]^\wedge_{\rho} e \xrightarrow{\sim} \frac{\hat{T}_F(\Sigma H^1)^*}{(m^*((\Sigma H^2)^*))}.$$

Proof. This is an application of Corollary 7.4.5.

For the purpose of concision in the conditions of the main theorems, we set up this

Definition 11.2.2. Given a profinite group $G$ and a residual semi-simplification $\rho$, a presentation datum for the moduli of representations of $G$ with residual semi-simplification $\rho$ is

- A basis for $\rho$ making the expression $\rho \cong \bigoplus_{i=1}^r \rho_i$ compatible with the ordering of the factors of the block diagonal subalgebra $\bigoplus_{i=1}^r M_d(F) \hookrightarrow M_d(F)$, where $\rho_i : E \to M_d(F)$.

- An $r$-pointed homotopy retract between $H$ and $C$, as in Example 7.2.3.

- A choice of $F$-algebra structure on $F(G)^\wedge_{\ker \rho}$ arising from choices of idempotents as in (7.3.3), compatible with the standard matrix idempotents in the codomains of the $\rho_i$.

11.3. Presentation of the moduli space $\text{Rep}_\rho$. Fix a residual semi-simplification $\rho$. Next we want to deduce a presentation for $\text{Rep}_\rho$. We will do this in the case that $\rho$ is multiplicity-free.

Because representations of $G$ parameterized by $\text{Rep}_\rho$ are continuous, it follows that the induced $F[G]$-action factors through $E := F[\wedge^\wedge G]$. Moreover, the condition that they have residual semi-simplification $\rho$ implies that they factor through the completion $E_{\ker \rho}^\wedge$.

(Note that one can assume that $\rho$ is multiplicity-free without any loss of generality on the algebras $E_{\ker \rho}^\wedge$ we study here.) The only difference from the previous subsection is that we now consider coefficients in $M_d(A)$ for commutative algebras $A \in \text{Aff}_F$, with its $F$-algebra structure arising from the choice of presentation datum. Formerly, we considered algebras in $\text{Aff}_F$.

We require one additional notion in order to state the presentation. Recall from §3.2 the notion of a simple cycle $\gamma$, the set $SC(r)$, and tensor of Ext$^1$-modules Ext$^1_{F[G]}(\gamma)$. Let

$$I_{\text{cyc}} \subset R \cong T_F^{\wedge} \Sigma \text{Ext}^1_{F[G]}(\rho, \rho)^* \cong T_F^{\wedge} \bigoplus_{1 \leq i, j \leq r} \Sigma \text{Ext}^1(\rho_j, \rho_i)^*$$

(note that this free $F$-algebra is not completed) denote the ideal generated by the submodule of “cyclic tensors”

$$\bigoplus_{\gamma \in SC(r)} \text{Ext}^1_{F[G]}(\gamma)^*$$
(where because we are using non-symmetric tensors, we sum over all of the simple closed paths \(SCP(r)\) that constitute the simple cycles \(SC(r)\)). Note that \(SCP(r)\) and \(SC(r)\) include the loop \(i \rightarrow i\) for each \(i \in r\).

Then let \(\check{T}_{\text{cyc}} \Sigma \text{Ext}^1(\rho, \rho)^*\) denote the “cyclic completion” of \(T_{\mathbb{F}} \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho)^*\) by \(I_{\text{cyc}}\), which admits an inclusion into \(\check{T}_{\mathbb{F}} \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho)\). Notice that the submodule

\[
m^* \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho) \subset \check{T}_{\mathbb{F}} \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho)
\]

lies within \(\check{T}_{\text{cyc}} \Sigma \text{Ext}^1(\rho, \rho)^*\). This is the case because any non-zero simple tensor of degree \(s\) on \(\text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho)^*\) includes at least \([s/r]\) simple cycles, and is expressible as the product of a cyclic tensor with a tensor of degree bounded by \(r\).

**Theorem 11.3.1.** Assume that \(G\) satisfies the \(\Phi_p\) finiteness condition. Assume also that the residual semi-simplification \(\rho\) is multiplicity-free. Fix a presentation datum (Definition 11.2.2). These choices induce

1. idempotents on the universal Cayley-Hamilton algebra \(E_D\) over \(D\) that define a GMA structure on \(E_D\) such that \(D^\text{GMA} = D^u\), and
2. a presentation \(\text{Rep}^\text{GMA}_{E_D} \cong \text{Spf} S^\text{GMA}_D\), where

\[
S^\text{GMA}_D := \check{S}_{\text{cyc}} \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho)^* / m^* \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho)^*
\]

(where the \(\mathbb{F}^*\)-algebra structure on \(S^\text{GMA}_D\) is forgotten \(\mathbb{F}\)-algebra in \(\text{Aff}_{\mathbb{F}}\)).

**Proof.** We begin with the \(\mathbb{F}\)-algebra structure on \(E^\wedge_\rho\) given by the isomorphism \(\rho^u : E^\wedge_\rho \sim M_d(\mathbb{F}) \otimes R\) of Theorem 11.2.1. The image under

\[
E^\wedge_\rho \longrightarrow E_D
\]

of the chosen idempotents of \(E^\wedge_\rho\), which lift the standard diagonal idempotents of \(\bigoplus_{i=1}^r M_d(\mathbb{F})\) via \(\rho\), induces a GMA structure on \(E_D\). Indeed, just like the idempotents of \(E_D\) supplied by [Che14, Thm. 2.22(ii)] in Theorem 10.3.3(1), they lift the standard idempotents of \(\mathbb{F}[G]/\ker \rho \cong E_D/\ker \rho \cong \bigoplus_{i=1}^r M_d(\mathbb{F})\). Therefore, by [Row88, Thm. 2.9.18(iii), pg. 242] (just as in the argument for [WWE18, Lem. 5.6.8]), there is some conjugation in \(E_D\) sending one ordered set of idempotents to the other. We also know that if one supplied a GMA structure, so do their conjugate. Thus we may take the image of the idempotents under \(\mathbb{F}[G]_{/\ker \rho}^u : E_D \rightarrow D^u\) as the idempotents of a GMA structure on \(E_D\). We have noted in Theorem 10.3.3(1) that the native pseudorepresentation \(D^u : E_D \rightarrow R_D\) is equal to \(D^\text{GMA} : E_D \rightarrow R_D\) induced by this GMA structure. This completes part (1).

We consider the auxiliary moduli functor \(\text{Rep}_{\mathbb{F}[G]_{/\ker \rho}}\), which sends \(A \in \text{Aff}_\mathbb{F}\) to homomorphisms

\[
\mathbb{F}[G]_{/\rho}^u \rightarrow M_d(A)
\]

that are compatible with the maps from \(\bigoplus_{i=1}^r M_d(\mathbb{F})\) into each of them. (The map to \(E^\wedge_\rho\) comes from the inverse of \(\rho^u\).) We claim that the map \(\mathbb{F}[G]_{/\rho}^u \rightarrow E_D\) induces an isomorphism of functors on \(\text{Aff}_\mathbb{F}\)

\[
\text{Rep}_{\mathbb{F}[G]_{/\rho}} \cong \text{Rep}^\text{GMA}_{E_D}.
\]

Indeed, because the residual semi-simplification of all representations parameterized by \(\text{Rep}_{\mathbb{F}[G]_{/\ker \rho}}\) is \(\rho\), Theorem 10.3.3 provides that these representations factor.
through $E_D$. Because of the compatibility of idempotents arranged above, these factorizations $E_D \to M_f(A)$ preserve the GMA structure. Therefore, each $A$-point of $\Rep_{\mathbb{F}[G]}$ induces an $A$-point of $\Rep_{E_D}$ via this factorization. There is a left inverse map $\Rep_{E_D} \hookrightarrow \Rep_{\mathbb{F}[G]}$ given by $\mathbb{F}[G] \to E_D$, which is also a right inverse because $\Rep_{E_D}$ and $\Rep_{\mathbb{F}[G]}$ admit compatible monomorphisms into $\Rep_{A}$.  

Now we apply Morita-equivalence of $\mathbb{F}[G]$ and $\mathbb{F}[G]$ to draw an isomorphism of functors on $\mathcal{A}$.

$$\Rep_{\mathbb{F}[G]} \cong \Rep_{\mathbb{F}[G]}$$

where $\Rep_{\mathbb{F}[G]}$ sends $A \in \mathcal{A}$ to $\mathbb{F}$-algebra homomorphisms

$$e\mathbb{F}[G] \to M_r(A).$$

(The same procedure is applied to GMAs in [BC09, §1.3.2].)

The augmented $\mathbb{F}$-algebra isomorphism of Theorem [11.2.1]

$$(e^\rho e)^{-1} : \hat{T}_\rho \Sigma H^1(G, \text{End}_G(\rho))^* \to \mathbb{F}[G]$$

allows us to calculate $\Rep_{\mathbb{F}[G]}$. Firstly we work in the case $\text{Ext}^2_{\mathbb{F}[G]}(\rho, \rho) = 0$ and consider idempotent-fixing homomorphisms

$$R \cong \hat{T}_\rho \Sigma H^1(G, \text{End}_G(\rho))^* \to M_r(A).$$

Using the universal property of $\hat{T}_\rho$, and writing $I_A \subset A$ for an ideal that is maximal among ideals of definition for $A$, we see that these homomorphisms correspond to the $A$-submodule of

$$\Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho) \otimes \bigoplus_{1 \leq i, j \leq r} \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho_i, \rho_j) \otimes A$$

that is the intersection of the kernels of the maps

$$f(\gamma, x) : \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho) \otimes A \to A/I_A$$

parameterized by $\gamma \in SC(r)$ and $x \in \text{Ext}^1_{\mathbb{F}[G]}(\rho)$. This condition ensures that the induced map $T_\rho \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho)^* \to M_r(A)$ factors through its completion at $I_{\text{cyc}}$, that is, $\hat{T}_{\text{cyc}} \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho)^*$.  

Next, admit the case that $\text{Ext}^2_{\mathbb{F}[G]}(\rho, \rho) \neq 0$. From the properties of free algebras, we deduce that $\Rep_{\mathbb{F}[G]}(A)$ is naturally isomorphic to

$$\text{Hom}_{\mathbb{F}^r}(\hat{T}_{\text{cyc}} \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho)^*, M_r(A)).$$

This in turn is naturally isomorphic to

$$\text{Hom}_{\mathbb{F}^r}(\hat{T}_{\text{cyc}} \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\rho, \rho)^*, A),$$

where we simply forget the $\mathbb{F}$-algebra structure on the domain. Finally, because $A$ is assumed to be commutative, we deduce the desired result.
Remark 11.3.2. One can derive from Theorem [11.3.1] the relationship established in [BC09, §1.5.3-1.5.4] between the structure of the GMA $E_D$ and various Ext-groups, mainly Ext$^1$. See especially [BC09, Rem. 1.5.7], which discusses the relationship with Ext$^2$: the ambiguities there are controlled by the $m^*$ map on Ext$^2_{\mathcal{F}[G]}(\rho, \rho)^\ast$.

11.4. Presentation of the pseudodeformation ring. Next we apply Theorem [10.3.3] in order to present the pseudodeformation ring $R_D$. We recall from Definition 3.2.4 the complete Noetherian local ring $R_D^1$, which will be shown to present $R_D$ when Ext$^2_{\mathcal{F}[G]}(\rho, \rho) = 0$.

**Theorem 11.4.1.** Assume that $G$ satisfies the $\Phi_p$ finiteness condition and assume that the residual semi-simplification $\rho$ is multiplicity-free. Fix a presentation datum (Definition 11.2.2).

These choices produce a presentation of $R_D$ as a complete Noetherian local $\mathbb{F}$-algebra with residue field $\mathbb{F}$,

$$
\begin{align*}
R_D^1 & \xrightarrow{R_D^1} R_D. \\
\bigoplus_{i,j \in \mathbb{F}} m^* \Sigma \text{Ext}^2_{G}(\rho_j, \rho_i)^\ast \otimes \left( \bigoplus_{\gamma \in \text{SCC}(i,j)} \Sigma \text{Ext}^1_{G}(\gamma)^\ast \right) & \xrightarrow{R_D^1} R_D.
\end{align*}
$$

**Proof.** The presentation for $R_D$ follows from combining

- the presentation for $S_{E_D}^{\Phi,\text{GMA}}$ of Theorem [11.3.1] with
- the result of [WE13, §2-3] stated in Theorem [10.3.3], that $R_D$ is the invariant subring of $S_{E_D}^{\Phi,\text{GMA}}$ under the adjoint action of the torus $Z(\rho)$.

Because $Z = Z(\rho)$ is linearly reductive over $\mathbb{F}$ (even in positive characteristic, as it is a torus), its invariant functor is exact. In particular, for any affine $\mathbb{F}$-scheme Spec $S$ that is a closed subscheme Spec $S' \cong \text{Spec} S'/I \subset \text{Spec} S'$ admitting a $Z$-action on $S'$ that preserves $S$, one has (see e.g. [AIP13, Rem. 4.11])

$$
S' = (S'/I)^Z \cong S'^Z/I^Z.
$$

We apply this to the presentation of $S_{E_D}^{\Phi,\text{GMA}} = S'/I$ as a quotient, where

$$
S' := \Sigma \text{Ext}^1_{\mathcal{F}[G]}(\rho, \rho)^\ast, \quad I := (m^* \Sigma \text{Ext}^2_{\mathcal{F}[G]}(\rho, \rho)^\ast);
$$

indeed, $I$ is $Z$-stable because it has generators that are isotypic for certain characters of $Z$ (these are in bijection with weights of the adjoint action of $Z$ as the torus in $\text{PGL}_r$).

We claim that $S'^Z \cong R^1_D$. Indeed, we see that a simple tensor in $S'$ is fixed by the adjoint action if and only if it is a cyclic tensor, and cyclic tensors generate precisely the image of (3.2.5) within the codomain $\Sigma \text{Ext}^1_{\mathcal{F}[G]}(\rho, \rho)^\ast$. Clearly the image is contained in the subring $\Sigma \text{cyc} \text{Ext}^1_{\mathcal{F}[G]}(\rho, \rho)^\ast$.

Similarly, we see that a generating set for $I^Z \subset I$ is formed as follows. Choose $\mathbb{F}$-basis $\{b_{i,j,k}\}$ for the generating vector space $m^* \text{Ext}^2_{\mathcal{F}[G]}(\rho, \rho)^\ast$ of $I$, such that its subset with fixed $(i,j)$ is a basis for $m^* \text{Ext}^2_{\mathcal{F}[G]}(\rho_j, \rho_i)^\ast$. Thus each $b_{i,j,k}$ is isotypic for the $Z$-action with the action depending only on $(i,j)$; call this character $\chi_{i,j}$. Then we observe that $I_Z$ is generated by $b_{i,j,k} \otimes c_{i,j,l}$, where for fixed $(i,j)$ the $c_{i,j,l}$ are a basis for a generating vector space of the $\chi_{i,j}^{-1} = \chi_{j,i}$-isotypic part of $S_{\text{cyc}} \Sigma \text{Ext}^1_{\mathcal{F}[G]}(\rho, \rho)$ as an $R^1_D$-module. The minimal such vector space is

$$
\bigoplus_{\gamma \in \text{SCC}(i,j)} \Sigma \text{Ext}^1_{\mathcal{F}[G]}(\gamma)^\ast.
$$
Then observe that this generating set \( \{ b_{i,j,k} \otimes c_{i,j,l} \} \) is a basis for the vector space in the denominator of (11.4.2).

Having presented \( R_D \), we can prove the rest of the results of §3.3 which are corollaries of the presentation. From now on, we refer to §3.3 for the statement of these corollaries.

We present the tangent space \( t_D \) of \( R_D \).

Proof of Corollary 3.3.5. Firstly we observe directly from the definition of \( R_D \) (Definition 3.2.4) that when \( \text{Ext}^2_{\mathbb{F}[G]}(\rho, \rho) = 0 \), then the presentation datum induces a presentation of its tangent space \( t_D \) as

\[
    t_D \cong \bigoplus_{\gamma \in SC(r)} \Sigma \text{Ext}^1_{\mathbb{F}[G]}(\gamma).
\]

Using the presentation of \( R_D \) in Theorem 11.4.1 and unraveling the definition of \( m^* \) in terms of the \( A_\infty \)-operations \( m_n \) from the bar equivalence (§5.4) for \( n \leq r \), we produce the maps of the statement of the corollary, whose common kernel is \( t_D \subset t_D \).

11.5. Canonical structure of the tangent space. As emphasized in Warning 3.3.6, the direct sum expression of \( t_D \) in Corollary 3.3.5 is highly non-canonical, being dependent on the presentation datum. The complexity filtration of [Bel12, §3] is a canonical filtration on \( t_D \) whose graded pieces are summands appearing in Corollary 3.3.5.

Definition 11.5.1 (Bellaïche). The complexity of a pseudodeformation \( D_A \) of \( D \) is the minimal integer \( c(D_A) \) such that for all \( \gamma \in SC(r) \) with length strictly greater than \( c(D_A) \), the image of \( \text{Ext}^1_{\mathbb{F}[G]}(\gamma)^* \) in \( R_D \) under the presentation (11.4.2) are sent to zero under the induced map

\[
    R_D \to A.
\]

Equivalently, the map of tangent spaces \( (m_A/m_A^2)^* \to t_D \) has image contained in

\[
    \text{Fil}_{c(D_A)}t_D := \bigoplus_{\gamma \in SC(r), l(\gamma) \leq c(D_A)} \text{Ext}^1_{\mathbb{F}[G]}(\gamma).
\]

The complexity filtration on \( t_D \) is the increasing exhaustive filtration consisting of the sums above. We denote it by \( \text{Fil}_k t_D \subset t_D \); observe that \( \text{Fil}_0 t_D = 0 \) and \( \text{Fil}_r t_D = t_D \).

Lemma 11.5.2. The complexity filtration of \( t_D \) is canonical.

Proof. To see that the complexity filtration is canonical, first observe that the ideal of cycles is a canonical ideal \( I_{\text{cyc}} \subset \mathcal{S}_E^{GMA} \). Indeed, it is \( I_{\text{cyc}} = m_D \cdot \mathcal{S}_E^{GMA} \), where \( m_D \subset R_D \hookrightarrow \mathcal{S}_E^{GMA} \) as the invariant subring of the \( Z \)-action (see the proof of Theorem 11.4.1). Then we have a decreasing filtration of \( \mathcal{S}_E^{GMA} \) given by \( I_{n,\text{cyc}}^n \), \( n \geq 0 \). The decreasing filtration on the cotangent space \( m_D/m_D^2 \) of \( R_D \) given by the intersection of \( m_D \) with \( I_{n,\text{cyc}}^n \) is perfectly dual to the complexity filtration of \( t_D \).

As we pointed out in Remark 3.3.9, the use of \( A_\infty \)-products or Massey higher products refines the result [Bel12 Thm. 1], which only used cup products. It results in a canonical determination of \( \text{gr}_k t_D := \text{Fil}_k t_D/\text{Fil}_{k-1} t_D \).
Proof of Corollaries 3.3.7 and 3.3.10. Exactly as in [Bel12, §3.3], there is a canonical injection
\[ \text{gr}_k t_D = \frac{\text{Fil}_k t_D}{\text{Fil}_{k-1} t_D} \hookrightarrow \bigoplus_{\gamma \in SC(r)} \Sigma \text{Ext}^1_F \left( \gamma \right). \]
We justify that this injection is canonical: there is a canonical identification of \( \ker(\rho) / \ker(\rho)^2 \cong \text{Ext}^1_F[\rho, \rho]^* \) (where \( \ker \rho \subset F[G] \)). This results is a canonical surjections
\[ (\text{Ext}^1_F[\rho, \rho]^*)^@n \twoheadrightarrow \ker(\rho)^n / \ker(\rho)^{n+1}. \]
The injection above results from taking the symmetric tensor quotient, then the invariants of the exact \( \mathbb{Z} \)-action, and then \( \mathbb{F} \)-linear duals.

Now Corollary 3.3.7 follows from the presentation (11.4.2) of \( R_D \) given in Theorem 11.4.1. Counting the dimensions of the vector spaces appearing in these formulas, we deduce Corollary 3.3.10. □

11.6. Input from invariant theory and quiver representation theory. Like the tangent dimension, the bounds on the Krull dimension of \( R_D \) claimed in Corollary 3.3.16 follow mostly from counting dimensions in the presentation of \( R_D \) of Theorem 11.4.1. The additional ingredient is our extra knowledge about the ring \( R^1_D \) from invariant theory. These have been stated in Fact 3.2.6. It is a summary of extensive literature about invariant subrings of regular rings under (linearly) reductive group actions. Overall, the point is that there are combinatorial objects controlling \( R^1_D \).

References for Fact 3.2.6. It is clear that \( R^1_D \) is reduced, as it is a subring of a domain. The fact that a subring of invariants of a regular commutative algebra under the action of a linearly reductive algebraic group is normal and Cohen-Macaulay is due to Hochster [Hoc72].

The decomposition into tensor factors, each arising from a strongly connected component, follows from the generation of \( R^1_D \) by cyclic tensors.

The claim about the Krull dimension is [LB90, Thm. 6]. There, the authors use quivers. The representation-unobstructed setting we are in can be translated to theirs by observing that the representation theory of \( \bar{T}_r \Sigma \text{Ext}^1_F[\gamma, \gamma]^* \) is the same as the representation theory of the quiver with \( r \) vertices labeled by \( \{1, \ldots, r\} \), and with \( h_{i,j}^1 \) directed arrows from \( i \) to \( j \). The dimension vector \( \alpha \) (in the notation of loc. cit. is \( \alpha = (1, \ldots, 1) \in \mathbb{N}^{\oplus r} \) (because each \( \rho_i \) appears with multiplicity 1 in \( \rho \) and the bilinear form \( R \) on \( \mathbb{Z}^{\oplus r} \times \mathbb{Z}^{\oplus r} \) due to Ringel is represented with the matrix
\[ R = (R_{i,j}) = \dim \text{Hom}_F[G](\rho_i, \rho_j) - \dim \text{Ext}^1_F[G](\rho_i, \rho_j) = \delta_{i,j} - h_{i,j}^1. \]
The we see in loc. cit. that the Krull dimension of \( R^1_D \) is \( R(\alpha, \alpha) \), which is equal to \( 1 - r + \sum_{1 \leq i,j \leq r} h_{i,j}^1 \), as claimed. The final claim follows from Krull dimension being additive under tensor products of commutative \( \mathbb{F} \)-algebras. □

Remark 11.6.1. It has been determined when \( R_D \) is regular [KKMSD73, §1.1, Thm. 4], complete intersection [Nak86] and Gorenstein [Sta78]. While Example 11.6.3 is complete intersection, it is rare for \( R_D \) to merely be Gorenstein in the representation-unobstructed case.
Remark 11.6.2. For a broader perspective on quiver representations in relation to this discussion, see [LL08 §§5.7-5.8], which relies on work of Bocklandt. The statement on Krull dimension is reproduced in [loc. cit., Lem. 5.13].

Example 11.6.3. There is a single simple cycle $\gamma = (12)$, and the only relation that must be imposed is the extra commutativity relation

\[(11.6.4) \quad (x_{12} \otimes x_{21}) \cdot (y_{12} \otimes y_{21}) = (x_{12} \otimes y_{21}) \cdot (y_{12} \otimes x_{21}),\]

which results in a quadratic obstruction of dimension $\binom{d_{12}}{2}$, $\binom{d_{21}}{2}$. In particular, $R_D$ is regular when either of $d_{12}$ or $d_{21}$ is $\leq 1$.

Now let’s consider the case $d_{12} = d_{21} = 2$. We can then take $\{x_{ij}, y_{ij}\} \subset \text{Ext}^1_G(\rho_j, \rho_i)^*$ to be a $k$-basis, so that the space of relations is 1-dimensional, generated by (11.6.4) as written. If we write

\[W = x_{12} \otimes x_{21}, \quad X = x_{12} \otimes y_{21}, \quad Y = y_{12} \otimes x_{21}, \quad Z = y_{12} \otimes y_{21},\]

then we readily see that

\[R_D \cong \frac{k[W, X, Y, Z]}{(WZ - XY)}.\]

Example 11.6.5. The singularity of Example 11.6.3 is the only possible singularity of $R^1_D$ when its Krull dimension is 3. For a classification of singularities in $R^1_D$ when its Krull dimension is $\leq 6$, see [BLBVdW03].

11.7. Obstruction theory. Here we prove the remaining statements of §3.3. All that is left to add is the following basic formulation of obstruction theory. While the obstructions $\beta$ are representation-theoretic, the obstructions $\alpha$ are combinatorial and can be calculated using the content of §11.6.

Proof of Fact 3.3.7. It is a standard fact that whenever $(S, m_S)$ is a regular local $F$-algebra surjecting onto $S'$ with kernel $I$, then the obstruction to lifting a homomorphism $\ell_n : S' \to F[\varepsilon]/(\varepsilon)^n$ to a homomorphism $S' \to F[\varepsilon]/(\varepsilon)^{n+1}$ is an element of $I/mI$ that can be produced from $\ell_n$. See, for example, [Maz89 §1.5, Prop. 2, pg. 399].

Then Fact 3.3.7 follows from observing that $h^2(C(D))$ as defined in Notation 3.2.2 is a basis for $I/mI$ in this case. Remark 3.3.12 follows from observing that $J$ of Notation 3.2.2 is finitely generated as a monoid.

Proof of Corollary 3.3.11. In light of the presentation for $R_D$ of Theorem 11.4.1 part (1) follows from Fact 3.3.7. Given that the obstruction $\alpha(D_n)$ of part (1) vanishes, Part (2) follows from the principle of Fact 3.3.7 the following computation principi. Choose $i, j \in r$ and element $\omega \in \text{Ext}^1_G(\rho_j, \rho_i)$. We have $m^*(\omega)$ in the $(i, j)$-part of $\hat{T}^*_F \cdot \text{Ext}^1_G(\rho, \rho)^*$. For each $\gamma \in \text{SCC}(i, j)$ and $k \in \text{Ext}^1_G(\gamma)^*$, $m^*(\omega) \otimes \kappa$ is an element of the $(k, k)$-part of $\hat{T}^*_F \cdot \text{Ext}^1_G(\rho, \rho)^*$. Therefore, taken as an element of $S_{\gamma} \cdot \text{Ext}^1_G(\rho, \rho)^*$, $m^*(\omega) \otimes \kappa$ is in the subring $R^1_D$. It was killed by $\psi^1_n$ (because $\psi^1_n$ exists), therefore $\psi^1_{n+1}(m^*(\omega) \otimes \kappa) \in \kappa^{n+1} \cong \mathbb{F}$. By duality, we see that $\psi^1_{n+1}$ gives rise to an element of the $F$-linear dual of the denominator of the presentation (11.4.2), as desired.

Remark 11.7.1. We expect that it is possible to give a representation-theoretic interpretation of the obstruction $\beta(D_n)$, using the fact (see e.g. [WWE17a §1]) that every pseudodeformation of $D$ comes from a GMA representation.
12. Galois representations satisfying arithmetic conditions

The goal of this section is to prove Theorem 3.4.1. This theorem claims that there exists a dg-algebra to which one can apply the theory of §11 to compute these deformation spaces with an extra condition $C$. Indeed, so far we have considered only the “unrestricted case,” presenting deformation spaces for representations of $G$.

The idea is straightforward: produce an algebra quotient $E_C$ of $F[[G]]$ that parameterizes exactly the representations with condition $C$. Then use the Hochschild cochain complex of the endomorphism module of a representation of this algebra to compute the deformations. Then, there is no need to recapitulate the proofs of §11 that establish the results claimed in §3.3: one simply uses $C^\bullet(E_C, \text{End}_F(\rho))$ in place of $C^\bullet(G, \text{End}_F(\rho))$.

12.1. Stable conditions and Cayley-Hamilton conditions. We begin by recalling previous results that construct the moduli spaces of representations of $G$ with condition $C$. We begin with a description of the conditions $\mathcal{C}$ that we will consider. Instead of working with particular conditions, we set up two different sorts of conditions to which we can apply our theorems.

In this particular section, we work with mixed characteristic coefficients: $\mathbb{Z}_p$ for simplicity.

**Definition 12.1.1** (Stable condition). A stable condition $C$ is a subcategory of finite length $\mathbb{Z}_p[G]$-modules that is closed under subquotients and finite direct sums.

Stable conditions first appeared in the study of Galois representations by Ramakrishna [Ram93], in the context of deformations rings of representations. When $\rho$ has scalar endomorphisms, there was produced a quotient $R_\rho \rightarrow R_C^\rho$ parameterizing exactly those deformations satisfying condition $C$.

In the author’s joint work with Preston Wake [WWE17a], stable conditions were shown to sensibly apply to Cayley-Hamilton representations, thereby allowing for

- a sensible definition of pseudorepresentations of a profinite group $G$ with condition $C$,
- a universal pseudodeformation ring $R_D^C$ parameterizing deformations of $D$ satisfying $C$, which cuts out a closed condition $R_D \rightarrow R_D^C$ in the whole deformation space,
- the construction of a Zariski-closed subspaces $\text{Rep}_{\rho}^{\Box, C}$, $\text{Rep}_{\rho}^{C}$ of representations with residual semi-simplification $\rho$ and condition $C$.

The other sort of condition $\mathcal{C}$ that we will consider does not sensibly apply to arbitrary finite length $F[G]$-modules; instead, the structure of a Cayley-Hamilton representation is required, or sometimes even a GMA representation. It does not seem worthwhile to produce a common intrinsic definition for the many such possible conditions. Instead, we make the following definition in terms of the key property that $\mathcal{C}$ must satisfy. For this we use the category of Cayley-Hamilton representations over a residual semi-simplification $\rho$; see [WWE17a Defn. 2.1.5].

**Definition 12.1.2.** Let $\rho$ be a residual semi-simplification of $G$. We say that $\mathcal{C}$ is a Cayley-Hamilton condition (over $\rho$) when it applies to any Cayley-Hamilton representation over $\rho$ and is “representable,” in the sense that there exists a universal object in the category of Cayley-Hamilton representations over $\rho$. We will denote such a universal object by $E_D^C$. 
We finish this section with an example of a Cayley-Hamilton condition. 

**Example 12.1.3** (Ordinary representations). A 2-dimensional representation of \( G_Q \) is called ordinary when its restriction to a decomposition group at \( p \) admits a 1-dimensional unramified quotient. See \[ \text{WE18} \S 7 \], as well as \[ \text{WWE18} \text{WE17a} \]. In each of these works, a “residually \( p \)-distinguished” condition was required in order to use the structure of a GMA on all Cayley-Hamilton representations. The condition was then that the GMA representation must be upper-triangular after restriction to a decomposition group at \( p \), with the quotient character being unramified. The main technical issue is to make sense of “upper-triangular” up to conjugation, in a GMA.

12.2. **Construction of a universal associative algebra with condition \( C \).** In this section, we propose a candidate associative algebra \( E_p^C \) which will play the role of \( \mathbb{F}[G] \) or \( \mathbb{F}[G]_{\ker \rho}^\wedge \) after imposing condition \( C \). In the case that \( C \) is a Cayley-Hamilton condition, the proposed algebra is \( E_p^C \). In the case of a stable condition \( C \), we generalize the construction of \[ \text{WWE17a} \] \S 2.4 \] in order to produce \( E_p^C \).

Let \( C \) be a stable condition. Notice that \( \mathbb{Z}_p[G] \) is a profinite algebra, which is a topological limit of its finite quotient algebras
\[
E(a, b) := \mathbb{Z}_p/\mathbb{Z}_p^a \mathbb{Z}_p[G_b],
\]
where \( G \cong \varprojlim \mathbb{Z}_p[G_b] \) is a profinite presentation for \( G \). Therefore, condition \( C \) may be sensibly applied to \( E(a, b) \), taking it as a left \( \mathbb{Z}_p[G]^C \)-module. By \[ \text{WWE17a} \text{WWE17a} \text{WWE17a} \text{Lem. 2.3.5} \], there is a maximal quotient module \( E(a, b) \to E(a, b)^C \) satisfying \( C \).

Therefore, for any \( a' \geq a, b' \geq b \), the \( \mathbb{Z}_p[G]^C \)-module quotient \( E(a', b') \to E(a, b)^C \) factors through \( E(a', b')^C \). Therefore we can produce a new limit
\[
E^C := \varprojlim_{a,b} E(a, b)^C.
\]

Now, as in \[ \text{WWE17a} \text{Lem. 2.4.3(2)} \], considering the the right action of \( \mathbb{Z}_p[G] \) on \( E(a, b) \to E(a, b)^C \) allows one to find that \( E(a, b)^C \) has the structure of an algebra quotient of \( E(a, b) \). Naturally, in the limit this makes \( E^C \) an algebra quotient of \( \mathbb{Z}_p[G] \).

When \( \rho \) has property \( C \), then \( \rho : E \to \text{End}_\mathbb{F}(V) \) factors through \( E^C \), and we let
\[
E^C_\rho := \varprojlim_i E^C / \text{ker}(\rho)^i.
\]

**Remark 12.2.1.** This generalizes \[ \text{WWE17a} \] \S 2.4 \] in that this construction was done while imposing the Cayley-Hamilton property there. All that is different is to notice that the finite length property of a quotient can be obtained more generally.

12.3. **Proof of Theorem 3.4.1.** Let \( C \) be a stable condition and let \( E^C \) be as constructed above. We choose a representation \( \rho \) of \( G \) with condition \( C \). Notice that we now have a natural inclusion of Hochschild cochain complexes
\[
C^*(E, \text{End}_\mathbb{F}(\rho)) \subset C^* (\mathbb{F}[G], \text{End}_\mathbb{F}(\rho))
\]
that is an inclusion of \( \text{dg-}\mathbb{F} \)-algebras. Choose compatible presentation data (as in Definition 11.2.2) for \( E \) and \( \mathbb{F}[G] \), meaning that the homotopy retract data \((i, p, h)\) on \( C^* (\mathbb{F}[G], \text{End}_\mathbb{F}(\rho)) \) restricts to that on \( C^* (E^C, \text{End}_\mathbb{F}(\rho)) \); and that the choices of idempotents of \( E^C_\rho \) and \( \mathbb{F}[G]^\wedge \) are compatible under \( \mathbb{F}[G] \to E^C \).

Write \( C^*_C \) for \( C^* (E^C, \text{End}_\mathbb{F}(\rho)) \), and write \( H^*_C \) for its cohomology. We now have a more specific statement of Theorem 3.4.1.
**Theorem 12.3.1.** Let $G$ be a profinite group satisfying finiteness condition $\Phi_p$, let $\rho$ be a multiplicity-free residual semi-simplification, and let $C$ be as above. The compatible presentation data above induce

- a presentation for $E^C_{\rho}$ as a $F_r$-algebra, exactly as in Theorem 11.2.1,
- presentations for $\text{Rep}^C_{GMA}$ and $\text{Rep}^C_{\rho}$ as formal moduli spaces over $\text{Spf} \, F$, exactly as in Theorem 11.3.1 and
- a presentation for $R^n_D$ as an object of $\hat{A}_F$, exactly as in Theorem 11.4.1

equipped with morphisms to (resp. from) the analogous unrestricted objects $F(G)_{\ker \rho}$, $\text{Rep}^C$, $\text{Rep}^\square_{\rho}$, and $R_D$. All are closed immersions (resp. quotients of rings).

Naturally, the formulas for the tangent space, representation-unobstructed case, obstruction theory, and Krull dimension also follow from the presentations, just as in §11.5 and §11.7.

The idea of the proof is simple: because $C \cdot C$ parameterizes exactly the functions on $G \times n$ which, when used to deform $\rho$, have property $C$, the deformations must have property $C$.

**Remark 12.3.2.** We discuss the contrast between the case that $C$ is a stable condition and $C$ is a Cayley-Hamilton condition over $\rho$ and $D = \psi(\rho)$. In order to compare them, it is instructive to consider the Cayley-Hamilton condition $C^{CH}$ arising from a stable condition, and the difference in the Hochschild complex for $E^C$ vs. $E^C_D$. It follows from the theorem that they will each compute exactly the same deformation space. It is also the case that $\text{Ext}^k(C,\rho_i) \cong \text{Ext}^k_{E^C_D}(\rho_j,\rho_i)$ for $k = 0, 1$. But there is a difference when $k = 2$. The classes of $\text{Ext}^2_{E^C_D}(\rho_j,\rho_i)$ are only those which appear as Massey products of representations with filtrations with graded pieces among the $[\rho_i]_{i=1}$. But $E^C$, having been constructed without any completion at $\ker \rho$, has the full $\text{Ext}^2$-groups of the subcategory $C$ of $F[G]$-modules.

### 12.4. Dg-algebra structures on suspended cones.

In number-theoretic applications, one often knows that deformations of a Galois representation satisfying a condition $C$ are controlled by Ext-functors, but they may not arise as group cohomology of the Galois group $G$. One common case is that these cohomology groups appear as a cone of better understood complexes, such as $G$ and distinguished subgroups. See e.g. [CHT08, pg. 22], or [WWE18, §6] for an ordinary case. The goal of this section is to set up a dg-algebra that is an alternative to the dg-algebra of functions on an associative algebra.

First we point out that a suspended cone of a dg-algebra morphism has a dg-algebra structure. This is similar to the analogous statement for dg-Lie algebras that appears in [FM07], but the associative case is simpler.

We work with a morphism of dg-algebras

$$\chi : L \to M.$$ 

We form the suspended cone

$$C_\chi := \text{Cone}(\chi)[-1], \quad d_{C_\chi}(l,m) = (d_LL, \chi(l) - d_MM).$$

In particular, $C_\chi = L^i \oplus M^{i-1}$, and the resulting exact triangle in the category of complexes of $\text{Vec}_F$ is

$$C_\chi \longrightarrow L \longrightarrow M.$$
**Proposition 12.4.1.** There is a natural dg-algebra structure on \( C_\chi \) such that the projection of complexes \( C_\chi \to L \) is multiplicative.

**Proof.** There is a natural multiplication on \( C_\chi \), expressed in coordinates as follows in terms of generic elements \( (l_i, m_i) \in C_\chi \), \( i = 1, 2 \).

\[
\begin{align*}
  l_1 \otimes l_2 &\mapsto l_1 \cdot l_2, \quad m_1 \otimes l_2 &\mapsto m_1 \cdot \chi(l_2) \\
  l_1 \otimes m_2 &\mapsto (-1)^{\deg_L(l_1)} \chi(l_1) \cdot m_2, \quad m_1 \otimes m_2 &\mapsto 0.
\end{align*}
\]

It satisfies Leibniz with respect to \( d_{C_\chi} \): we compute the less obvious compatibilities

\[
\begin{align*}
  d_{C_\chi}((0, m_1) \cdot (l_2, 0)) &= d_{C_\chi}(m_1 \cdot \chi(l_2)) = -d_M(m_1 \cdot \chi(l_2)) \\
  &= -d_M m_1 \cdot \chi(l_2) - (-1)^{\deg_M(m_1)} m_1 \cdot \chi(d_M l_2) \\
  &= d_{C_\chi}(0, m_1) \cdot (l_2, 0) + (-1)^{\deg_{C_\chi}(0, m_1)} (0, m_1) \cdot d_{C_\chi}(l_2, 0)
\end{align*}
\]

(Using that \( \deg_M(m_1) = \deg_{C_\chi}(0, m_1) - 1 \) and)

\[
\begin{align*}
  d_{C_\chi}(l_1,0) \cdot (0, m_2) &= -d_M((-1)^{\deg_L(l_1)} \chi(l_1) \cdot m_2) \\
  &= (-1)^{\deg_L(d_M l_1)} \chi(d_M l_1) \cdot m_2 - \chi(l_1) \cdot d_M m_2 \\
  &= d_{C_\chi}(l_1,0) \cdot (0, m_2) + (-1)^{\deg_{C_\chi}(l_1)} (l_1,0) \cdot d_{C_\chi}(0, m_2)
\end{align*}
\]

(where the sign of the second term of the second line arises from \( \deg_M(\chi(l_1)) = \deg_L(l_1) \)).

This multiplication is also associative. We calculate that both

\[
(l_1, m_1) \cdot ((l_2, m_2) \cdot (l_3, m_3)) \quad \text{and} \quad ((l_1, m_1) \cdot (l_2, m_2)) \cdot (l_3, m_3)
\]

are equal to

\[
(l_1 \cdot l_2 \cdot l_3, m_1 \cdot \chi(l_2 \cdot l_3) + (-1)^{\deg_L(l_1)} \chi(l_1) \cdot m_2 \cdot \chi(l_3) + (-1)^{\deg_L(l_1 \cdot l_2)} \chi(l_1 \cdot l_2) m_3).
\]

□

In order to apply the main theorem, the following observation is useful.

**Lemma 12.4.2.** Let \( \chi : L \to M \) be a map of dg-F-algebras, and let \( C_\chi \to L \) be the suspended cone with its dg-F-algebra map to \( L \). Given choices of homotopy retracts

\[
L \leftarrow H^*(L), \quad M \leftarrow H^*(M)
\]

that are compatible in the sense that the homotopy retract maps \((i, p, h)\) commute with \( \chi \), there exists a compatible homotopy retract

\[
C_\chi \leftarrow H^*(C_\chi).
\]

**Proof.** Take the direct sum of the two retract structures, as expressed in Example 5.2.2. □

13. Ranks of \( p \)-adic Hecke algebras

In this section, we give examples of \( p \)-adic completions (or interpolations) \( T \) of classical Hecke algebras acting on modular forms. These are known to be free of finite rank over a regular local complete Noetherian \( \mathbb{Z}_p \)-algebra. Here, we give an expression for this rank in terms of \( A_\infty \)-products. This is a measure of the size of the module of congruent modular forms. It would be interesting to relate this quantity to some sort of analytic invariant, i.e. an appropriate modulo \( p \) version of an adjoint \( L \)-function.
The approach is to use a known isomorphism $R \sim \mathbb{T}$ where $R$ is some deformation ring of Galois representations, and then use $A_\infty$-products to calculate the rank of $R$.

We cover four cases.

- The finite-flat residually non-Eisenstein setting of Wiles [Wil95],
- the ordinary residually non-Eisenstein setting of Wiles *op. cit.*,
- the ordinary Eisenstein setting of Ribet’s converse to Herbrand’s theorem [Rib76], and
- the finite-flat residually Eisenstein setting of Mazur’s Eisenstein ideal [Maz77], following [WWE17c].

“Finite-flat” and “ordinary” are conditions on Galois representations. The residually Eisenstein/non-Eisenstein distinction is more serious, because on the Galois side this is the residually irreducible/reducible distinction.

Let $G = G_{\mathbb{Q},S}$ be the Galois group of $\mathbb{Q}$ with ramification only at a finite set of primes $S$, where $S$ is the support of $Np\infty$, where $N \geq 1$. Let $\ell \mid N$ be a prime, and let $G_\ell \to G$ be a decomposition group for $\ell$.

13.1. Weight 2 non-Eisenstein non-ordinary Hecke algebras. In this section and the next, we discuss the Hecke algebras proved to be isomorphic to a Galois deformation ring by Wiles [Wil95]. Here, we focus on the finite-flat case, which is case (ii) on [pg. 456, *loc. cit.*]. Write $\rho$ here for the residual representation

$$\rho : G_{\mathbb{Q},S} \to \text{GL}_2(\mathbb{F})$$

written $\rho_0$ in *loc. cit.*; in particular, it is absolutely irreducible. We assume that $\rho$ is ramified at all $\ell \in p$. Assume for simplicity that $\rho|_{I_{\ell}}$ satisfies cases (B) or (C) (but not (A)) of [pg. 458, *loc. cit.*]. This makes for a deformation condition denoted $D$ there, which includes the finite-flat condition on $G_p$. We let $G$ be the maximal quotient of $G_{\mathbb{Q},S}$ in which $\ker(\rho|_{I_{\ell}}) \subset I_{\ell}$ vanishes. Then, let $E_D$ be the maximal quotient of $\mathbb{F}[G]$ that is finite-flat upon restriction to $G_p$, in the sense of [12.2]

**Proposition 13.1.1.** *The Hochschild cochain complex* $C_D := C^\bullet(E_D, \text{End}_\mathbb{F}(\rho))$ *calculates the deformation ring* $R_p/pR_D$ *of* [Wil95] *pg. 458* *via Theorems 3.1.1* *and 12.3.1*. *In particular, the* $A_\infty$-*products in* $H^\bullet(C_D)$ *determine the rank of* $R_D \cong \mathbb{T}$.

**Proof.** Indeed, because $\rho$ is irreducible, one has $r = 1$ and $S^\rho\text{-GMA} \cong R_D$ represents $\text{Rep}_\rho$. Because $R_D$ is free of finite rank over $W(\mathbb{F})$ by the $R_D \sim \mathbb{T}$ theorem of [Wil95] Thm. 3.3, the $\mathbb{F}$-dimension of $R_D/pR_D$ is equal to this rank. This $\mathbb{F}$-dimension can be calculated using the provided presentation. \hfill \Box

Using [Wil95] (1.5), pg. 460, we find that $H^1(C_D)$ is canonically isomorphic to the $p$-torsion of the “adjoint Selmer group” $H^1_\mathbb{Z}(\mathbb{Q}, \text{End}_\mathbb{F}(\rho))$ given there. The $A_\infty$-products or Massey products $m_n : H^1(C_D) \otimes^n \to H^2(C_D)$ are obstruction classes in the category of finite-flat left $\mathbb{F}[G]$-modules.

**Example 13.1.2.** For instance, if $\dim_{\mathbb{F}} H^1(C_D) = 1$, we know that the rank is at least 2, i.e. there is some non-trivial congruence between modular forms. Then $R_D/pR_D \cong \mathbb{F}[e_n]$ for some $n \geq 1$. The rank of $\mathbb{T}$ is then $n + 1$. Let $a \in H^1(C_D)$ be a basis. Then $n$ is the greatest $i \geq 1$ such that $m_i(b^{\otimes 1}) = 0$ in $H^2(C_D)$. This cohomology class also can be calculated as a Massey power (Definition 8.2.1).
13.2. **Ordinary non-Eisenstein Hecke algebras.** We now assume that \( \rho \) is as in \([13.1]\) but now in case (i) on [pg. 456, loc. cit.]. We form \( G' \) exactly as we defined \( G \) in \([13.1]\) to take account of conditions at the places \( S \setminus \{p\} \). Next, we take the universal Cayley-Hamilton quotient \( E \) in \([13.1]\) \( \mathbb{G} \) and then its ordinary Cayley-Hamilton quotient \( E_D \) of \([WWE18] \S 5\) (see Example \([12.1.3]\)). This matches the ordinary condition (i-b) on [pg. 457, loc. cit.].

Let \( E_D \) be the further Cayley-Hamilton quotient with some fixed determinant valued in \( \mathbb{W} \); denote the corresponding deformation ring by \( R_D \). (This is essentially the “Selmer” deformation condition of (i-a) on [pg. 456, loc. cit.].) We claim that we can express this as a stable condition, at least for representations with residual semi-simplification \( \rho \). For this, we require the supplementary hypothesis that \( \chi_1 \chi_2 \) has order at least 3, the Jordan-H"{o}lder factors \( \chi_1, \chi_2 \) of \( \rho|_{G_p} \). The stable condition is given as follows. Let \( \chi_{1,A} : G_p \to \mathbb{F}[[t]] \) be any unramified deformations of \( \chi_1 \), to \( A \in \mathbb{C}_p \), and let \( e_A \) be any extension of \( \chi_{2,A} \) by \( \chi_{1,A} \). Let \( H \subset G_p \) be the intersection of the kernels of the \( G_p \)-action on all such extensions. Then we observe that a deformation of \( \rho|_{G_p} \) is ordinary if and only if \( H \) is in its kernel. Thus we may express the “ordinary with fixed determinant” condition by considering deformations of \( \rho \) that kill \( H \): let \( G \) be the maximal quotient of \( G' \) in which \( H \) vanishes. Let \( E_D := \mathbb{F}[G']_p \).

In this setting, \( R_D \cong \mathbb{T} \) is a “big” ordinary Hecke algebra, originally constructed by Hida. This is a finite rank free \( \Lambda \)-module, where \( \Lambda \cong \mathbb{W}[[t]] \) and \( \Lambda \to \mathbb{T} \) is the weight map induced by the inclusion of the Hecke algebra of diamond operators. Equivalently, \( \Lambda \to \mathbb{T} \) parameterizes the determinant, so \( R_D \cong R_D \otimes_{\Lambda} \mathbb{W} \) for a map \( \Lambda \to \mathbb{W} \) corresponding to the fixed determinant. So we the \( \Lambda \)-rank of \( R_D \cong \mathbb{T} \) is equal to the \( \mathbb{F} \)-dimension of \( R_D/pR_D \).

**Proposition 13.2.1.** The Hochschild cochain complex \( C_D := C^*(E_D, \text{End}_F(\rho)) \) calculates the deformation ring \( R_D/pR_D \) of \([Wli95] \) pg. 458 via Theorem \([12.3.1]\). Similarly, the Hochschild cochain complex \( C_D := C^*(E_D', \text{End}_F(\rho)) \) calculates the deformation ring \( R_D'/pR_D' \), and so determines the \( \Lambda \)-rank of \( R_D \cong \mathbb{T} \).

As in \([13.1]\) one can check that \( H^1(C_D) \) is canonically isomorphic to the reduction modulo \( p \) of the ordinary \( H^1_S(\mathbb{Q}, \text{End}_F(\rho)) \) of \([Wli95] \) (1.5), pg. 460]. While \( H^2(C_D) \) will calculate the right thing, the category \( \mathcal{E} \) for which it computes an \( \text{Ext}^2_{\mathbb{F}}(\rho, \rho) \) is somewhat limited; see Remark \([12.3.2]\).

13.3. **Ordinary residually Eisenstein Hecke algebras.** We follow \([WWE18] \S 2\) (and the references therein) for the setting of ordinary modular forms we work with. We work in the case \( S = \{p, \infty\} \), i.e. we work with ordinary modular forms of level 1, for simplicity. It is possible to work with more general level, but we leave this out so that the relevant hypothesis is precisely Vandiver’s conjecture.

Let \( \omega \) denote the modulo \( p \) cyclotomic character. The relevant residual semi-simplification is \( \rho \cong \omega^{k-1} \oplus 1 \) over \( \mathbb{F}_p \), where \( 2 \leq a \leq p - 3 \) is even. Ribet \([Rib76]\) proved that there is a non-trivial \( \omega^{1-k} \)-part \( A[\omega^{1-k}] \) of the \( p \)-cotorsion \( A \) of the class group of \( \mathbb{Q}(\zeta_p) \) if and only if \( p \) divides the numerator of the Bernoulli number \( B_k \). Ribet’s idea was to use a modular eigenform produce a \( \mathbb{F}_p \)-linear Galois representation of the form

\[
\begin{pmatrix}
\omega^{k-1} & 0 \\
* & 1
\end{pmatrix} : G_{Q,S} \to \text{GL}_2(\mathbb{F}_p),
\]
which is not semi-simple, but is semi-simple after restriction to $G_p$. Let $\mathbb{T}$ be the Hida Hecke algebra with residual eigensystem corresponding to $\rho$. It is a finite free $\Lambda$-algebra (in the same fashion as in \[13.2\]) admitting a homomorphism $\mathbb{T} \to \Lambda$ corresponding to the interpolation of ordinary $p$-stabilizations of Eisenstein series of level 1 and weight congruent to $k$ modulo $p - 1$. Assume also that $p \mid B_k$, so that $\mathbb{T}$ has rank at least 2, reflecting that it parameterizes some cusp forms with a congruence with these Eisenstein series.

Upon the assumption that $A[\omega^{-k}]$ is zero — which follows from Vandiver’s conjecture, as $-k$ is even — there is proved in [WWE17b, Thm. 4.2.8] an isomorphism

$$R^D_D \cong \mathbb{T},$$

where $R^D_D$ is an ordinary pseudodeformation ring for $D := \psi(\rho)$, as in \[12.1\]. Because $\omega^{-k}$ is not of order 2, the “ordinary with fixed determinant $\omega^{-k}$” condition can be expressed as a stable condition, just as in \[13.2\]. We let $\mathcal{D}$ be this deformation condition, in equal characteristic $p$. Under this assumption, the $\text{Ext}^1$-group for the Cayley-Hamilton condition $\mathcal{D}$ (writing $\chi = \omega^{-k}$ for convenience) is

$$\text{Ext}^1_{\mathcal{D}}(\rho, \rho) \cong \begin{pmatrix} \text{Ext}^1_{\mathcal{D}}(\chi, \chi) & \text{Ext}^1_{\mathcal{D}}(1, \chi) \\ \text{Ext}^1_{\mathcal{D}}(\chi, 1) & \text{Ext}^1_{\mathcal{D}}(1, 1) \end{pmatrix},$$

and one can compute that

$$\text{Ext}^1_{\mathcal{D}}(\chi, \chi) \cong \text{Ext}^1_{\mathcal{F}[G_{\omega,s}]}(\chi, \chi)^{I_p,\text{triv}} \cong H^1(G_{\mathbb{Q},s}, \mathbb{F}_p)^{I_p,\text{triv}} \cong 0,$$

$$\text{Ext}^1_{\mathcal{D}}(1, \chi) \cong \text{Ext}^1_{\mathcal{F}[G_{\omega,s}]}(1, \chi) \cong H^1(G_{\mathbb{Q},s}, \chi), \quad \dim_{\mathbb{F}_p} = t$$

$$\text{Ext}^1_{\mathcal{D}}(\chi, 1) \cong \text{Ext}^1_{\mathcal{F}[G_{\omega,s}]}(\chi, 1)^{G_p,\text{triv}} \cong H^1_{(p)}(G_{\mathbb{Q},s}, \chi^{-1}) \cong A[\chi^{-1}]^*, \quad \dim_{\mathbb{F}_p} = s$$

$$\text{Ext}^1_{\mathcal{D}}(1, 1) \cong \text{Ext}^1_{\mathcal{F}[G_{\omega,s}]}(1, 1)^{I_p,\text{triv}} \cong 0.$$
Due to the isomorphism $R_D^\text{ord} \cong \mathbb{T}$ and the fact that $R_D^\text{ord} \cong R_D^\text{ord}/m_AR_D^\text{ord}$, we know that $R_D^\text{ord}$ has Krull dimension 0, and that its $\mathbb{F}_p$-dimension is the $\Lambda$-rank of $\mathbb{T}$. In particular, we know that the injection of (13.3.1) is an isomorphism.

It is expected that $s = 1$; this is a consequence of the assumption $A[\omega^k] = 0$, which follows from Vandiver’s conjecture but is different than our running assumption that $A[\omega^{-k}] = 0$. In this case, the presentation for $R_D^\text{ord}$ is of the form $\mathbb{F}_p[c_n]$. Here $n$ may be computed as follows. Let $c = c_1$.

**Proposition 13.3.2.** The Hochschild cochain complex $C_D := C^*(E_D, \text{End}_\mathbb{F}(\rho))$ calculates the deformation ring $R_D$ via Theorem [12.3.1].

Assume

1. Vandiver’s conjecture, and
2. assume that $p \mid B_k$, so that the $\Lambda$-rank of $\mathbb{T}$ is at least 2 and $s \geq 1$.

Then we may read off the presentation for $R_D$ that the $\Lambda$-rank of $\mathbb{T}$ is equal to $n + 1$ and $\mathbb{T}/m_\Lambda \mathbb{T} \cong \mathbb{F}[c_n]$, where $n \geq 2$ is the greatest such that

$$m_{2n-3}(c \otimes b \otimes c \otimes \cdots \otimes b \otimes c) = 0.$$ 

**Remark 13.3.3.** As a complement to the discussion above, we see that a failure of Vandiver’s conjecture in the form of the existence of $p$ and $k$ such that

1. $t > 1$, when $s \geq 1$ as well; or
2. $s > 1$

is detectable by the tangent dimension of $R_D^\text{ord}$ being greater than 1. See [Wak15 §1.2] for a discussion of known relationships between Vandiver’s conjecture (and its consequences) and the Gorenstein property of Hecke algebras. It is interesting to compare these, because we might guess that $R_D^\text{ord} \sim \mathbb{T}$ even when the hypothesis $A[\omega^{-k}] = 0$ fails.

13.4. The finite-flat residually Eisenstein Hecke algebra of Mazur. In this example of a Hecke algebra, we give an example of a Cayley-Hamilton condition that cannot be expressed as a stable condition, but nonetheless has deformation theory controllable by Massey products with defining systems chosen to match the Cayley-Hamilton condition, according to [WWE17c] Thm. 1.3.1.

Let $\mathbb{T}^0$ be the Hecke algebra arising from the Hecke action on weight 2 level $\Gamma_0(N)$ cusp forms with a congruence with the Eisenstein series modulo $p$, where $N$ is a prime. Mazur proved that $\mathbb{T}^0 \neq \mathbb{T}$ if and only if $p$ divides the numerator of $(N - 1)/12$, and asked for an expression for the rank of $\mathbb{T}^0$ [Max77 §19].

In [WWE17c] Thm. 1.3.1, an expression is given in terms of Massey products in Galois cohomology for the $\mathbb{Z}_p$-rank of $\mathbb{T}^0$. This theorem relies on an isomorphism $R_D^C \cong \mathbb{T}$, where $\mathbb{T} \to \mathbb{T}^0$ is the cuspidal quotient of the full Eisenstein-congruent Hecke algebra $\mathbb{T}$, and $\text{rank}_{\mathbb{Z}_p} \mathbb{T} - \text{rank}_{\mathbb{Z}_p} \mathbb{T}^0 = 1$. We will now explain the Cayley-Hamilton condition condition $C$ that determines $R_D^C$.

In this setting, the residual semi-simplification is the representation $\rho = \omega \oplus 1 : G_{\mathbb{Q}, S} \to \text{GL}_2(\mathbb{F}_p)$, where $S = \{p, N, \infty\}$. Let $D = \psi(\rho)$ be the induced pseudorepresentation. We want to study the deformations of $\rho$ and $D$ satisfying $C$, where $C$ is the combination of the following two conditions:

1. the finite-flat condition upon restriction to $G_p$, which is a stable condition (in the sense of Definition [12.1.1]), and
2. the Cayley-Hamilton condition that the restriction to $G_N$ induces a trivial pseudodeformation on $I_N$; see [WWE17c Defn. 10.1.2]. We will call this condition “pseudo-unramified at $N$.”
The Massey products of [WWE17c Thm. 1.3.1] are valued in \( H^2(G_{\mathbb{Q}, S}, \text{End}_{\mathbb{F}_p}(\rho)) \) and are given in terms of defining systems of lifts of \( \rho \) (to coefficients in \( \mathbb{F}_p[\epsilon_n] \)) that are chosen iteratively: see [WWE17c §10]. This is basically the same process as the iteratively constructed lifts and Massey powers in §9.3. These defining systems are designed so that the lifts satisfy \( C \).

In contrast with the previous three examples, it does not seem that the pseudo-unramified at \( N \) condition can be imposed as a stable condition on deformations of \( \rho \).

However, we can apply our theory to the part of condition \( C \) that is stable, that is, the finite-flat part. This gives a formulation of the finite-flat Galois cohomology referred to in [WWE17c Rem. 10.6.3]. In particular, it values the Massey products in a cohomology group \( H^2_{\text{flat}}(G_{\mathbb{Q}, S}, \text{End}_{\mathbb{F}_p}(\rho)) \) realized as the Hochschild cohomology we now define. Namely, just as in §13.1, we let \( \mathcal{E}_{\text{flat}} \) be the maximal quotient of \( \mathbb{F}_p[G_{\mathbb{Q}, S}] \) that is finite-flat upon restriction to \( G_p \).

**Proposition 13.4.1.** The cohomology groups

\[
H^i_{\text{flat}}(G_{\mathbb{Q}, S}, \text{End}_{\mathbb{F}_p}(\rho)) := H^i(\mathcal{E}_{\text{flat}}, \text{End}_{\mathbb{F}_p}(\rho)).
\]

satisfy the desiderata of finite-flat cohomology of [WWE17c Rem. 10.6.3] and support an \( A_\infty \)-algebra structure quasi-isomorphic to the dg-algebra \( C^\bullet(\mathcal{E}_{\text{flat}}, \text{End}_{\mathbb{F}_p}(\rho)) \).

Then the Massey product computations of [WWE17c §10] can be repeated in the cochain complex \( C^\bullet(\mathcal{E}_{\text{flat}}, \text{End}_{\mathbb{F}_p}(\rho)) \). When this is done, the finite-flat condition on a deformation will be automatic, when the Massey product vanishes. This contrasts with how the arguments of loc. cit. must arrange for an unrestricted deformation to be “adjusted” so that it becomes finite-flat. It remains that the pseudo-unramified-at-\( N \) condition must be arranged for by hand.

Note that it was possible for the purposes of [WWE17c] to make the aforementioned adjustment and forgo constructing this finite-flat cohomology because it was understood that once it was produced, there would be an injection

\[
H^2_{\text{flat}}(G_{\mathbb{Q}, S}, \text{End}_{\mathbb{F}_p}(\rho)) \hookrightarrow H^2(G_{\mathbb{Q}, S}, \text{End}_{\mathbb{F}_p}(\rho)),
\]

so that the vanishing of Massey products could be calculated in unrestricted Galois cohomology without additional complications. However, this does not always hold for similar deformation problems of interest, so statements like Proposition 13.4.1 are useful.

**References**


DEFORMATIONS OF RESIDUALLY REDUCIBLE GALOIS REPRESENTATIONS

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