1. Recall the definition of a function of bounded variation. Suppose f is of bounded variation on $[0,T]$. Let λ be a real number. Is λf of bounded variation on $[0,T]$? If f, g are of bounded variation on $[0,T]$ is $f+g$ of bounded variation on $[0,T]$?

2. Prove that a monotone increasing function, defined on $[0,T]$, is of bounded variation on $[0,T]$. Let $h = f - g$ where f and g are monotone functions on $[0,T]$. Is h of bounded variation on $[0,T]$?

3. Are all functions defined on $[0,T]$ of bounded variation on $[0,T]$?

4. Justify your answer to 3.

5. Recall the definition of Riemann-Stieltjes integration. Let $f(x) = I_{[0,\frac{1}{2}]}(x)$. Does

$$\int_0^1 f(x)\,df(x)$$ exist?

6. Suppose that g is continuous on $[0,T]$ and f is differentiable there. Does

$$\int_0^T g(x)\,df(x)$$

exist? Can you write $\int_0^T g(x)\,df(x)$ as an "ordinary" integral when it does exist?

7. Let E be a collection of subsets of $\mathbb{R}$. Prove that;

(i) There is a σ-field on $\mathbb{R}$ that contains E.

(ii) If $\{S_\lambda : \lambda \in \Lambda\}$ is a collection of σ-fields each of which contains E then, $\cap\{S_\lambda : \lambda \in \Lambda\}$ is a σ-field containing E.

(iii) If $\{S_\ell : \ell \in L\}$ is the collection of all σ-fields on $\mathbb{R}$ that contain E then, $\cap\{S_\ell : \ell \in L\}$ is the smallest σ-field on $\mathbb{R}$ containing E. (You have say what is meant by "smallest".)

8. Let E be the collection of bounded intervals in $\mathbb{R}$. So a typical $J \in E$ has one of the forms: (a,b), $[a,b)$, $(a,b]$, $[a,b]$, for some $a, b \in \mathbb{R}$ with $a \leq b$.

Let $F(E)$ denote the collection of sets which are finite unions of disjoint elements of E. Prove that $F(E)$ is not a σ-field.

9. Let E be as in 8 and $\sigma(E)$ the smallest σ-field on $\mathbb{R}$ containing E. Let $\mathbb{Q}$ denote the rational numbers. Prove that $\mathbb{Q} \in \sigma(E)$.

This is a discussion not a question.

10. For an element, G, of $F(E)$ (see 8.) with, $G = J_1 \cup J_2 \cup \ldots \cup J_k$, say, and $J_i \cap J_\ell = \emptyset$ if $i \neq \ell$, define

$$\lambda(G) = \sum_{i=1}^{k} \lambda(J_i)$$

where $\lambda(J_i)$ = length of J_i, that is, if $J_i = (a_i, b_i]$, say then $\lambda(J_i) = b_i - a_i$, and similarly if J_i is one of the other kinds of intervals.

It doesn't matter how you write G as a union of J's from E $\lambda(G)$ is always the same number ($\lambda(G)$ is well-defined ....). Also, if we take the union of an increasing sequence from $F(E)$, say,

$$G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n \subseteq G_{n+1} \subseteq \ldots$$

Then by defining $\lambda(\bigcup_n G_n) = \lim_n \lambda(G_n)$ we get a well-defined measure for $\bigcup_n G_n$. Similarly if we have a decreasing sequence: $G_i \in F(E)$ and

$$G_1 \supseteq G_2 \supseteq \ldots \supseteq G_n \supseteq G_{n+1} \supseteq \ldots$$

then $\lambda(\bigcap_n G_n) = \lim_n \lambda(G_n)$ gives a well-defined measure to $\bigcap_n G_n$. Staying with this last case we can learn something.

11. Cantor's Set. Let $I_0 = [0,1]$. Remove from I_0 the "open middle third", $(1/3, 2/3)$, and call the remainder I_1. Then $I_1 = [0, 1/3] \cup [2/3, 1]$. For each interval in I_1, remove its open middle third; call the remainder I_2. Then $I_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 9/9]$.

Before we proceed let's summarise:

We began with one closed and bounded interval, I_0. We removed the open middle third; this left us with two closed bounded intervals. We removed the open middle thirds from each interval in I_1. This left us with four closed bounded intervals whose union is I_2.

$$\lambda(I_0) = 1, \quad \lambda(I_1) = 2/3, \quad \lambda(I_2) = 4/9$$

$$I_0 \supseteq I_1 \supseteq I_2.$$

We repeat this procedure ad-infinitum. More precisely, we suppose that we have defined $I_0, I_1, \ldots, I_{n-1}$. That for $j = 0, 1, \ldots, n-1$, I_j is a union of 2^j disjoint closed intervals with $I_0 \supseteq I_1 \supseteq \ldots \supseteq I_{n-1}$ and $\lambda(I_j) = \left(\frac{2}{3}\right)^j$.

We obtain I_n by removing from each of the 2^{n-1} closed bounded intervals that make up I_{n-1}, the open middle third of that interval. I_n is what remains of I_{n-1} after each middle third has been removed. So we see that $I_n \subseteq I_{n-1}$, I_n comprises 2^n closed bounded intervals (each interval in I_{n-1} "spawns" two intervals of I_n). These intervals are disjoint (because those of I_{n-1} were disjoint...). $\lambda(I_n) = \left(\frac{2}{3}\right)^n$ because we have removed a set of length $\frac{1}{3}$ of $\lambda(I_{n-1})$ and $\lambda(I_{n-1}) - \frac{1}{3}\lambda(I_{n-1})$ is $\frac{2}{3}\lambda(I_{n-1}) = \frac{2}{3}\left(\frac{2}{3}\right)^{n-1}$.

Is this going somewhere? Luckily, the answer is yes.

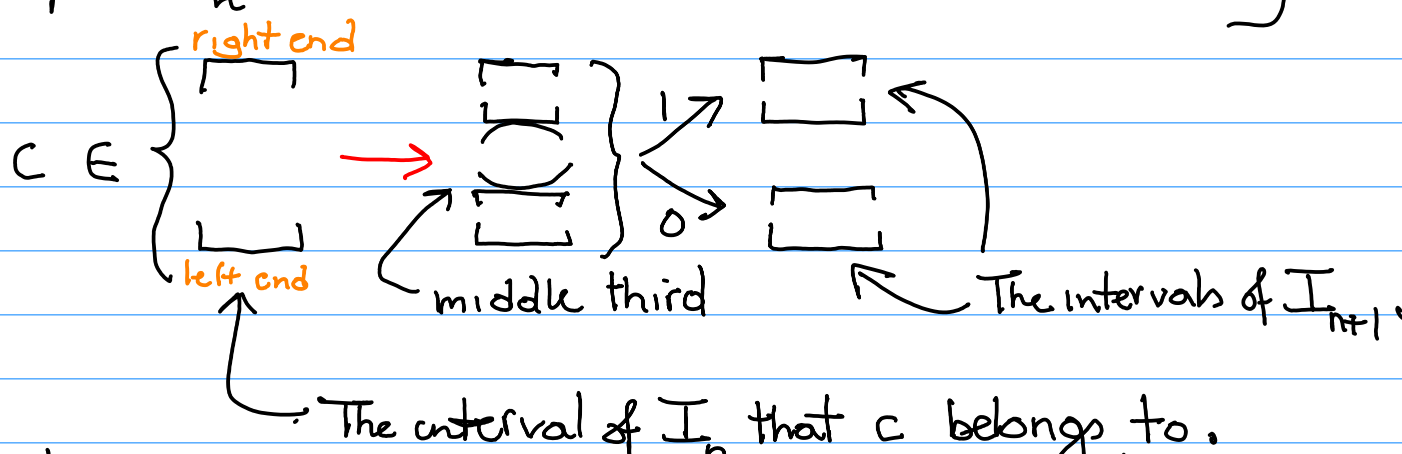
Consider $\bigcap_{n\geq 0} I_n$. We can assign a measure to this set.

$$\lambda\Big(\bigcap_{n\geq 0} I_n\Big) = \lim_n \lambda(I_n) = 0$$

$\triangleq$ means "I name this ship"

So $\bigcap_{n\geq 0} I_n \triangleq C$ is a set of zero measure. Is it empty? No!, 0, 1, 1/3, 2/3, ... are in C — check this for yourself and while you are at it notice that if x is an end-point of one of the intervals of I_n then $x \in I_k$ for all $k \geq n$ and is therefore in C.

Suppose $c \in C$. Then $c \in I_n$ for each $n \geq 0$. Now I_n comprises 2^n disjoint closed intervals. So c belongs to just one of these intervals. As the construction of I_{n+1} proceeds each interval of I_n "spawns" two (disjoint) intervals of I_{n+1} and if we consider the interval of I_n that contains c then c can belong to just one of the two intervals of I_{n+1} that are spawned by the interval of I_n that contains c. Pictorially



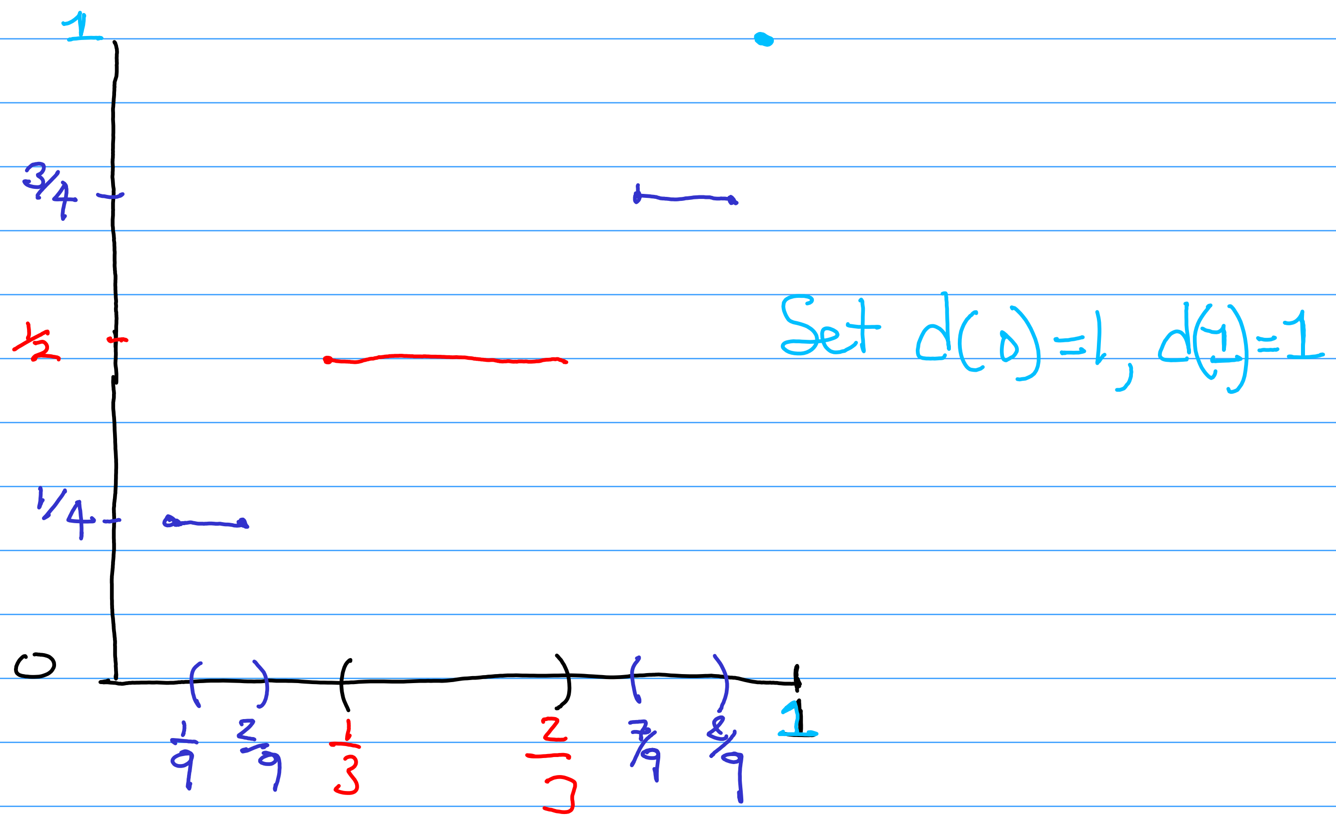
I have stood the interval on its end, so $[a,b]$ is written (and I left out the end points too). I've also written 1 to indicate that c belongs to the "upper third" of and wrote 0 to indicate that c belongs to the "lower third" Of course only one of these possibilities is true. This gives us a way of describing c. For sure, $c \in I_0$, so lets write 1 to indicate this. Now $c \in I_1$ and we write 1 if c belongs to the upper third and 0 if it belongs to the lower third. Continuing in this way we obtain a binary sequence for c. If c_1 and c_2 are distinct elements of C then their binary sequences must differ because for sufficiently large n $|c_1 - c_2| > \frac{1}{3^n}$ and this is the length of the intervals of I_n that c_1 and c_2 belong to. Remember they can belong to just one of these, so to be in the same interval of I_n would require $|c_1 - c_2| \leq 1/3^n$. So at some point their binary sequences must differ. So each $c \in C$ has a unique binary sequence that describes its "journey" through the intervals of $I_0, I_1, I_2, \ldots, I_n, \ldots$. On the other hand. If we choose any binary sequence, prefix it with 1 and interpret it as describing the decreasing sequence of closed bounded intervals that contains some point of C then it uniquely identifies the single point in their intersection, (Cantors Intersection Theorem). So the points of C and these binary sequences are in 1 to 1 correspondance. We know the binary sequences are uncountable and so, therefore, are the elements of C.

12.

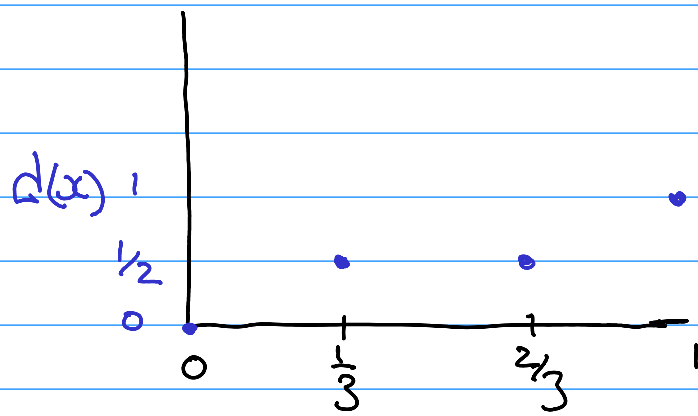
REM Sets of Probability (measure) zero are quite important. The fact that there is a probability space, $([0,1], \sigma(E), \lambda)$, with uncountable sets of measure zero means we have to take extra care with our arguments.

13.

Define a function, d, on $[0,1]$ as follows: $d(0) = 0$, $d(1) = 1$. On $(1/3, 2/3)$ set $d(x) = \frac{1}{2}$ and on $(7/9, 8/9)$ set $d(x) = 3/4$ while on $(1/9, 2/9)$ it is set to be $1/4$. Perhaps a picture helps:



We construct d by following the construction of C. Consider I_0. We already know the values of d at each of the end-points of I_0. Moving to I_1, we remove the open middle third of I_0 and define d on the closed middle third to be halfway between the values of d at the left and right endpoint of I_0. That is, $d(x) = \frac{1}{2}(f(0)+f(1)) = \frac{1}{2}$ on $[1/3, 2/3]$. Returning to the intervals of I_1; $[0, 1/3]$, $[2/3, 1]$, observe that we know the value of d at each of the end points of each of the intervals of I_1.



We can see now that we know the values of d at each of the end points of every interval of I_1. One more step should make the procedure clear. Moving to I_2 we remove the open middle third of each interval in I_1. We define d on each closed middle third to be halfway between the values of d at the end-points of the interval of I_1 from which the open third has been removed. We see that now we know the values of d at the end-points of every interval that comprises I_2 and we also know the value of d on every open third removed up to this stage.

So let's suppose we have followed the procedure outlined above and we have the set I_k to deal with next. We assume also that we know the values of d at the end points of every closed bounded interval that makes up I_k and the values of d on every open third removed so far (put another way we know d on the gaps between the intervals of I_k). Let $[u,v]$ be an interval in I_k. We know $d(u)$ and $d(v)$. Define d on $\left[u+\frac{(v-u)}{3}, v-\frac{(v-u)}{3}\right]$ (the closed middle third) to be $\frac{1}{2}\left(d(u)+d(v)\right)$.

Having done this for every interval of I_k we now know the values of d at every end point of every interval of I_{k+1} and the value of d on every open third removed so far.

By iterating this construction define $d(x)$ on every interval of $[0,1]$

which is removed in the construction of C. Now, any point of C, x, say, is the intersection of a unique decreasing sequence of closed bounded intervals — recall our demonstration that C is uncountable. We know the value of d at each end-point of each interval of this sequence whose intersection is x. Writing $[a_n, b_n], n \geq 0$, for this decreasing sequence of intervals and noting that $[a_n, b_n]$ is one (and only one) of the intervals of I_n, that the restriction of d to the end-points of the intervals of I_n is an increasing function and that $d(b_n) - d(a_n) \leq 1/2^n$, then by defining

$$d(x) = \lim_n d(a_n) = \lim_n d(b_n)$$

we can extend d to all of $[0,1]$.

14. Prove that d is well-defined on all of $[0,1]$.

15. Prove that d is monotone increasing.

16. Prove that d is continuous on $[0,1]$.

REM: Use the construction above for 14, 15, 16. If you are easy with liminfs and limsups you can get nice elegant demonstrations.

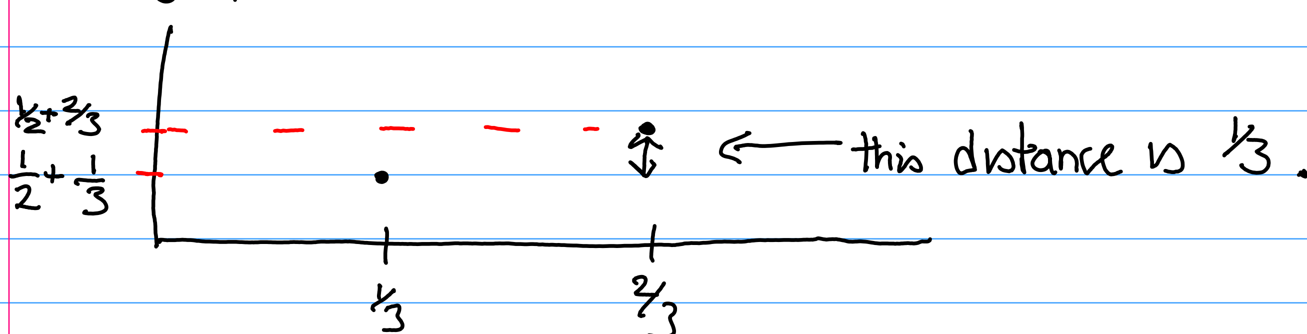
17. C has measure zero. So $[0,1]\setminus C$, which is the sum of the lengths of all of the middle thirds removed from $[0,1]$, must be 1. Verify this directly.

18. On each middle third, d is constant. Therefore d is differentiable on each middle third and $d'(x) = 0$ there. By defining $d'(x) = \infty$ on C confirm that,

$$\int_0^1 f'(x)\, d\lambda(x) = 0 \neq f(1) - f(0).$$

19. <u>Discussion</u>

Let $ds(x) = d(x) + x$. Then ds is a continuous function. Clearly $ds(0) = 0$ and $ds(1) = 2$. Let us consider ds restricted to $[1/3, 2/3]$. We know that $d(x) = 1/2$ on $[1/3, 2/3]$ so that $ds(1/3) = d(1/3) + 1/3$ and $ds(2/3) = d(2/3) + 2/3$ and the graph of $ds(x)$ looks like this on $[1/3, 2/2]$:



So ds maps the closed middle third into an interval of length $1/3$. A moments thought tells us that since d is constant on each

closed middle third arising from the removal of an open middle third in the construction of C, then each closed middle
closed* third is mapped to an interval* of the same length by ϕ. So $[0,1]\setminus C$ is mapped to a set which is the union of a sequence of disjoint intervals the sum of whose lengths is 1. It follows that ϕ maps C to a set whose measure is 1! So here is a continuous function which maps a set of zero measure to a set of positive measure. You can check that ϕ is strictly increasing and continuous and that its inverse is continuous too. The function $\phi^{-1}(x)$ maps a set of positive measure to a set of measure zero.

20. Let h be a function on $\mathbb{R}$ which is continuous. Prove that $h^{-1}(E)$ is open whenever E is open in $\mathbb{R}$.
Prove that every open set is Borel Measurable.
Prove that every closed set is Borel Measurable.
Is it true that $h^{-1}(G)$ is closed whenever G is closed in $\mathbb{R}$?
Consider the collection; $\{F \subseteq \mathbb{R} : h^{-1}(F) \text{ is a Borel Set}\}$. Investigate the structure of this Collection: Is it closed under unions?, intersections?, complementation? ....

21. I will give you this exercise only if you have completed 1. to 20, you are dying to know how the story ends and you have little else to do.

22. Let our probability space be $([0,1], \bar{\mathcal{B}}, \lambda)$, that is, $\Omega = [0,1]$, $\bar{\mathcal{B}}$ is the Borel Lebesgue σ-field and λ is Lebesgue measure. Let $f(x) = \dfrac{1}{2\sqrt{x}}$ for $0 < x \leq 1$ and $f(0) = 0$.

Prove that

$$\bar{\mathcal{B}} \ni E \longmapsto \int_E f(x)\,dx$$

is a probability measure, $\mathbb{Q}$, on $\bar{\mathcal{B}}$. Is $\mathbb{Q}$ equivalent to λ? Can you find a function which is λ integrable but not $\mathbb{Q}$ integrable? Are $L^\infty([0,1], \bar{\mathcal{B}}, \lambda)$ and $L^\infty([0,1], \bar{\mathcal{B}}, \mathbb{Q})$ 'the same'? Is every $\mathbb{Q}$ integrable function also λ integrable? Are there a pair of constants, m, M, say, such that, $m > 0 < M$,

$$m\,\|h\|_1^{\lambda} \leq \|h\|_1^{\mathbb{Q}} \leq M\,\|h\|_1^{\lambda}. ?$$

Reflect upon this example when we use the random variable,

$$e^{\int_0^t h(s)\,dW_s - \frac{1}{2}\int_0^t h(s)^2\,ds}$$

to change measures in Mathematical Option Pricing.

23. (Don't do this one)

Define an equivalence relation on $\mathbb{R}$,
$u \sim v \iff u - v \in \mathbb{Q}$:
Prove that $\sim$ is symmetric, reflexive and transitive.
For $x \in \mathbb{R}$, write $[x] = \{y \in \mathbb{R} : x \sim y\}$ for the equivalence class of x under $\sim$.
So, for each $y \in [x]$ we have $x = y + q$ where $q \in \mathbb{Q}$ and, conversely, any y satisfying this equation for a rational q must lie in $[x]$.

Prove that the distinct equivalence classes of $\sim$ partition $\mathbb{R}$.

Prove that $[x]$ is countable, deduce that there are an uncountable number of equivalence classes of $\sim$, and $[x]$ is dense in $\mathbb{R}$.

So no two distinct elements of V are equivalent

Use the Axiom of Choice to choose from each equivalence class an element which belongs to $[0,1]^{\dagger}$. Call the set of all such elements V (for Vitali!).

Is V measurable? If it is then any translation of V by an element, h, of $\mathbb{R}$, i.e., $V + h = \{v + h : v \in V\}$, is measurable too, (exercise). So let $q_1, q_2, \ldots$
$\ldots q_k, \ldots$ be an enumeration of

$\mathbb{Q} \cap [-1,1]$ and write $V_k = V + q_{v_k}$.

Prove that $V_k \cap V_\ell = \emptyset$ if $k \neq \ell$.

Prove that $[0,1] \subseteq \bigcup_k V_k$.

Prove that $V_k \subseteq [-1,2]$ for each k.

Deduce that

$$1 \leq \sum_{k=1}^{\infty} \lambda(V_k) \leq 3,$$

while $\lambda(V) = \lambda(V_k)$ for each k, and that V cannot be measurable.

24. Don't do this one.

So, 23. shows that not all sets are measurable. The axiom of choice was used to construct one inside of $[0,1]$. In fact you can find a non-measurable set inside of any set whose (outer) measure is positive.

Return for a moment to 20. Some of you may have shown, some of you may already know, that the inverse image of a Borel set, E, under a continuous function, h, is once again a Borel Set. So the function $ds(x)$, which is continuous and with a continuous inverse (it's a homeomorphism). So now consider $\phi(C)$, C is the Cantor Set. This has positive measure and therefore contains a non-measurable set, E, say. The set $\phi^{-1}(E)$ is a subset of C and is therefore measurable in the Lebesgue sense. But it cannot be measurable in the Borel sense because its inverse image under ϕ^{-1} — $(\phi^{-1})^{-1}(E)$ — would have to be a Borel set were this true. But E is not measurable and therefore certainly not Borel measurable. So, not a measurable sets are Borel sets. One can show that the Borel sets are the same size as $\mathbb{R}$, but the measurable sets are of the order of $2^{\mathbb{R}}$. ($\aleph_1$ and $\aleph_2$??).

25. (X_t) is a process, adapted to the Stochastic Base $(\Omega, (\mathcal{F}_t), \mathbb{P}, [0,T])$, with $X_t \in L^1(\mathcal{F}_t)$ and, $\forall t$ $\mathbb{E}(X_t)$ is a constant. Is (X_t) necessarily a martingale? Examples please.

26. (X_t) is an L^2-martingale on $(\Omega, (\mathcal{F}_t), \mathbb{P}, [0,\infty))^{\dagger}$ with the property that,

$\dagger$ Usual Conditions

$$\sup_t \|X_t\|_2 < \infty.$$

By considering $\|X_t - X_s\|_2^2$ for $s<t$, prove

that $t \mapsto \|X_t\|_2^2$ is increasing and that $\forall \epsilon > 0 \ \exists t_\epsilon \in \mathbb{R}^+$ with $\|X_t - X_s\|_2 < \epsilon$ wherever $t, s \geq t_\epsilon$. Prove that there is $X \in L^2(\mathcal{F}_\infty)$ such that

$$X_t \to X \quad \text{in } L^2 \text{ norm}$$

and

$$M_t^{\mathbb{P}}(X) = X_t .$$

Why doesn't this work for (B_t), Brownian Motion? What does happen to (B_t) as $t \to \infty$? Does it converge in any sense?

Dividend Model, Hyperfinite time line.

$\frac{0 \cdot T}{N}, \frac{1 \cdot T}{N}, \ldots \left(\frac{N-1}{N}\right)T,$ N is an infinite integer.

At $\frac{0}{N}$ we hold (ϕ_0, ψ_0). At $\frac{T}{N}$ we receive and amount of wealth $\frac{\delta \phi_0 S_0}{N}$. Which we deploy, $\frac{\rho_0 \delta \phi_0 S_0}{N}$ into S and $\frac{\delta(1-\rho_0)\phi_0 S_0}{N}$ into B. Here ρ_0 can be understood as a (generalised) proportion of the wealth received which is commuted into S at time T/N. So the stock and bond numbers derived from this allocation are, respectively,

$$\frac{\rho_0 \delta \phi_0 S_0}{N S_{\frac{T}{N}}} \quad \text{and} \quad \frac{\delta(1-\rho_0)\phi_0 S_0}{N B_{\frac{T}{N}}}$$

(ϕ_0, ψ_0) is rebalanced in a self-financing manner to $(\phi_{\frac{T}{N}}, \psi_{\frac{T}{N}})$. So the numbers for $\frac{T}{N}$ to $\frac{2T}{N}$ are

$$\phi_1 + \frac{\rho_0 \delta \phi_0 S_0}{N S_{\frac{T}{N}}} \text{ of } S, \; \psi_1 + \frac{\delta(1-\rho_0)\phi_0 S_0}{N B_{\frac{T}{N}}} \text{ of } B.$$

At $\frac{2T}{N}$ we receive dividend cash amounting to

$$\delta\left(\phi_1 + \frac{e_0 \delta \phi_0 S_0}{N S_{\frac{T}{N}}}\right) S_{\frac{T}{N}} \cdot \frac{1}{N}$$

which we deploy as

$$e_1 \delta\left(\phi_1 + \frac{e_0 \delta \phi_0 S_0}{N S_{\frac{T}{N}}}\right) S_{\frac{T}{N}} \cdot \frac{1}{N} \cdot \frac{1}{S_{\frac{2T}{N}}} \quad \text{of } S$$

$$(1-e_1)\,\delta\left(\phi_1 + \frac{e_0 \delta \phi_0 S_0}{N S_{\frac{T}{N}}}\right) S_{\frac{T}{N}} \cdot \frac{1}{N} \cdot \frac{1}{B\left(\frac{2T}{N}\right)} \quad \text{of } B, \text{ and}$$

we rebalance (ϕ_1, ψ_1) to (ϕ_2, ψ_2).

Take stock: The numbers for S over $\frac{2T}{N}$ to $\frac{3T}{N}$

$$\phi_2 + \frac{e_1 \delta \phi_1 S_{\frac{T}{N}}}{N \cdot S\left(\frac{2T}{N}\right)} + \frac{e_1 e_0 \delta^2 \phi_0 S_0}{N^2 S\left(\frac{2T}{N}\right)}$$

and the numbers for B are

$$\psi_2 + \frac{\delta(1-e_0)\phi_0 S_0}{N B(T/N)} + \frac{(1-e_1)\delta \phi_1 S\left(\frac{T}{N}\right)}{N B\left(\frac{2T}{N}\right)} + \frac{(1-e_1) e_0 \delta^2 \phi_0 S_0}{N^2 B\left(\frac{2T}{N}\right)}$$

At $3T/N$ we get $e_2 \delta\left(\phi_2 + \frac{e_1 \delta \phi_1 S(T/N)}{N \cdot S\left(\frac{2T}{N}\right)} + \frac{e_1 e_0 \delta^2 \phi_0 S_0}{N^2 S\left(\frac{2T}{N}\right)}\right) S\left(\frac{2T}{N}\right) \frac{1}{N\, S\left(\frac{3T}{N}\right)}$

items of S and

$$(1-e_2)\,\delta\left(\phi_2 + \frac{e_1 \delta \phi_1 S(T/N)}{N \cdot S(2T/N)} + \frac{e_1 e_0 \delta^2 \phi_0 S_0}{N^2 S(2T/N)}\right) \frac{S\left(\frac{2T}{N}\right)}{N} \cdot \frac{1}{B(3T/N)} \quad \text{of } B$$

The numbers for S and B over $\frac{3T}{N}$ to $\frac{4T}{N}$ are....
(ϕ_2, ψ_2) is rebalanced to (ϕ_3, ψ_3) ....

$$\phi_3 + \frac{\rho_2 \delta \phi_2 S(\frac{2T}{N})}{N\, S(\frac{3T}{N})} + \frac{\rho_2 \rho_1 \delta^2 \phi_1 S(T/N)}{N^2 S(\frac{3T}{N})} + \frac{\rho_2 \rho_1 \rho_0 \delta^3 \phi_0 S_0}{N^3 S(\frac{3T}{N})}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\phi_k + \frac{\rho_{k-1} \delta \phi_{k-1} S(\frac{(k-1)T}{N})}{N\, S(\frac{kT}{N})} + \frac{\rho_{k-1} \rho_{k-2} \delta^2 \phi_{k-2} S(\frac{(k-2)T}{N})}{N^2 S(\frac{kT}{N})} + \frac{\rho_{k-1} \rho_{k-2} \rho_{k-3} \delta^3 \phi_{k-3} S(\frac{(k-3)T}{N})}{N^3 S(\frac{kT}{N})}$$

$$+ \cdots + \frac{\rho_{k-1} \rho_{k-2} \cdots \rho_2 \rho_1 \rho_0 \delta^k \phi_0 S_0}{N^k S(\frac{kT}{N})}$$

$$\rho_2 \frac{\delta \phi_2 S_2}{N S_3} + \rho_2 \rho_1 \frac{\delta \phi_1 S_2}{N^2 S_3} + \frac{\rho_2 \rho_1 \rho_0 \delta^2 \phi_0 S_0}{N^3}$$