Stopping Times : Familiarisation Exercises

1. Let $\Omega = \mathbb{B}_n = \{ \omega = (\omega_i) \in \mathbb{R}^n : \omega_i \in \{0,1\} \}$, ie. the set of binary n-tuples in $\mathbb{R}^n$.

   Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, \Omega, E_1, E_0\}$ where $E_1 = \{\omega : \omega_1 = 1\}$, $E_0 = \{\omega : \omega_1 = 0\}$. So for $1 \leqslant k \leqslant n$, $\mathcal{F}_k = \sigma\{ E_{\epsilon^k} : \epsilon^k \in \mathbb{B}_k \}$, here $E_{\epsilon^k} = \{\omega : (\omega_1, \dots, \omega_k) = \epsilon^k\}$ ie those ω's that are identical with a particular ϵ^k in their first k places. Set $\mathcal{F}_\infty = \mathcal{F}_n$.

   Let $\tau : \Omega \to \{1, 2, 3, \dots, n, \infty\}$ be defined as

   $$\tau(\omega) = \inf\{ i : \omega_i = 1\}$$

   with, as usual, $\inf \emptyset = \infty$. (τ is the first time a 1 appears)

   Consider the event $\pi_k^1 = \{\omega : \omega_k = 1\}$. Prove that this is an $\mathcal{F}_k$ set. And prove the same for its brother π_k^0. Identify $\{\tau \leqslant k\}$ with some combin-ation of π_i^r's, $1 \leqslant i \leqslant k$, $r \in \{0,1\}$. Deduce that τ is a stopping time of the filtration $(\mathcal{F}_i)_{i=1}^{i=\infty}$.

   Clearly this would serve as a model for a finite sequence of tosses of a coin. What is not so obvious is that it also serves as the mathematics on which one can build a binomial stock.

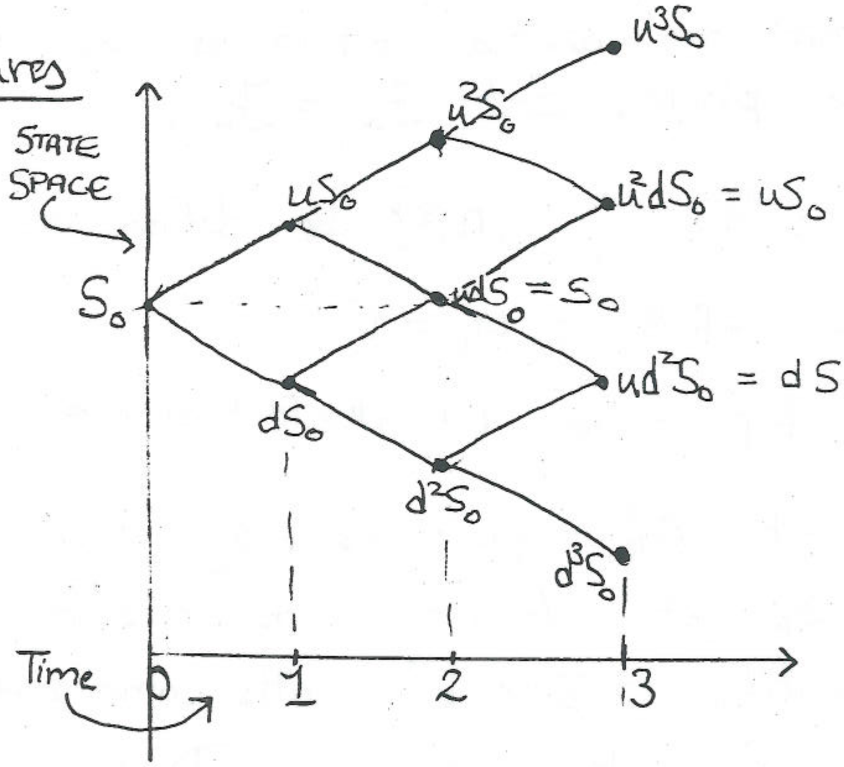
2. We biuld a binomial stock model on $(\Omega, \mathcal{F}_n)$ in question 1 with the following definition. For $0 \leq k \leq n$,

$$\mathbb{R}^+ \ni S_0 > 0$$

$$S_k = S_{k-1}\left(u I_{\pi_k^1} + d I_{\pi_k^0}\right)$$

here $\mathbb{R} \ni u > 1$ and $\mathbb{R} \ni d < 1$, both are positive. Often one takes $d = \frac{1}{u}$, but you don't have to.

<u>Pictures</u>



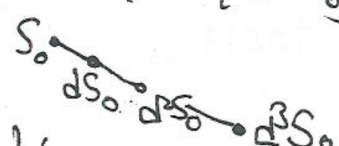
A "recombining tree" of stock prices $(d = 1/u)$

Here $n=3$ and our Ω is just 3-tuples of 0's and 1's. Choose a certain value of S, say dS_0, ask the question "when is the first time $S_k(\omega) = dS_0$?" Look at each <u>path</u> of S "through" the "tree" above and work out when this time occurs. S_0, for example, the path

<u>never</u> hits dS_0 so, for this path,

$$\min\{i : S_i(\omega) = dS_0\} = \tau(\omega) = \infty,$$

because $\{i : S_i(\omega) = dS_0\}$ is empty. On the other hand the <u>path</u>



clearly hits dS_0 at time 1. Given an exhaustive answer to the question posed above.

2 continued.

Another (equivalent) way of writing down S is this; set

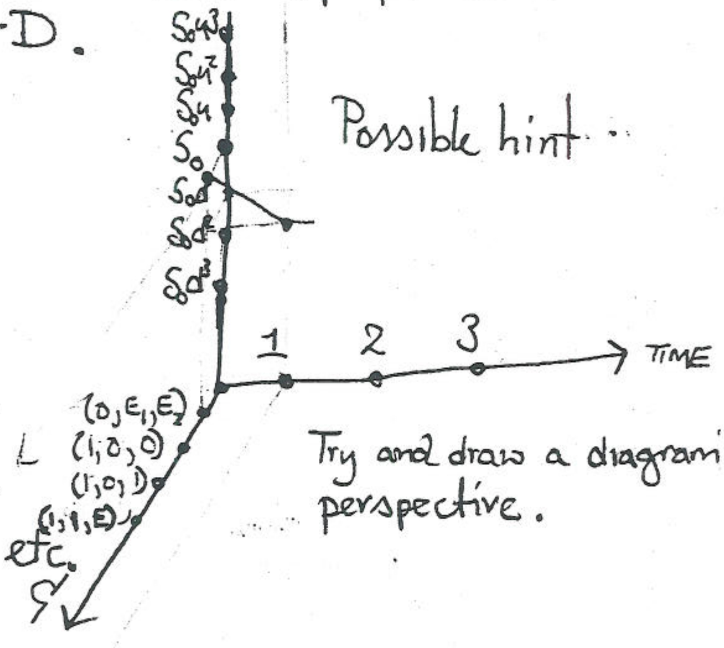
$$\Delta_t(\omega) = \sum_{i=1}^{t} \omega_i = \#\text{ of } 1\text{'s in } \omega \text{ upto "time } t\text{"}.$$

Then $S_t(\omega) = S_0 u^{\Delta_t(\omega)} d^{t-\Delta_t(\omega)}$, if we say $\Delta_0(\omega) = 0$ this defines $S_t(\omega)$ for $0 \leq t \leq n$. If we stipulate that $d = \frac{1}{u}$ and ask the question; "for which t is it true that $S_t(\omega) = S_0 d$?" Then this amounts to, "for which t is it true that $d = u^{\Delta_t(\omega)} d^{t-\Delta_t(\omega)}$?". Answer this question in terms of ω. Hint find an expression for "the number of 1's minus the # of 0's up to time t". The point of this exercise is to relate the "first time" that S exhibits some behaviour to the "first time" that the ω's exhibit some corresponding behaviour.

3. Draw, somehow, the process $S^\tau = (S_{\tau \wedge t})$ where $\tau = \min\{i : S_i(\omega) = S_0 d\}$. I think you might need lots of different coloured pens for this. And three or four time steps will be enough. Perhaps one of those charts you can produce in Excel or StarOffice might be appropriate. The drawing in Question 2 is not appropriate here! You need something 3-D.

Picture

Possible hint...



Try and draw a diagram with perspective.