Tenor-Varying Barrier Structures

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Abstract

By using numerical integration we price a novel form of barrier structure which is closely related to a barrier swaption. The new feature is where the barrier is monitored ***before and during*** the exchange arrangement and the arrangement can cease at any 'reset date'. We call this a Tenor-Varying Barrier Structure.

1 Introduction to Tenor Varying Barrier Structures

We would like to introduce a new type of barrier structure. It is similar to a barrier swaption but differs in two ways. The structure is an option on an exchange arrangement between two parties. Key features of the *arrangement* determined at the outset are;

- A barrier level, $H > 0$,
- An underlying which we call (K_t), which is the P almost surely strictly positive semimartingale $(K_0 e^{\sigma B_t + (r - \frac{\sigma^2}{2})t})$ with $K_0 < H$.
- A fixed payment, k.
- A *Tenor structure* which is a finite sucession of dates, $0 < T_0 < T_1 < \ldots < T_n$. The dates, T_{i-1} $1 \leq i \leq n$, are *reset dates* and T_i $1 \leq i \leq n$ are *payment dates*.

1.1 The Arrangement

First of all we consider the exchange arrangement without any barrier structure in place. This is almost exactly the same as a swap. The arrangement begins at time T_0. Over each of the time periods, $[T_{i-1}, T_i]$, $1 \leq i \leq n$, one party will pay the fixed amount k the other will pay an amount determined by K. The amount determined by K is set in advance, that is, it is $K_{T_{i-1}}$, (the reset date) but it is paid in arrears, that is, at T_i (the payment date). The party paying the fixed amount, k is by convention "the payer" and the other "the receiver". The narrative takes the view of the payer. The value at time $t \leq T_0$ of this arrangement, to the payer, is

$$e(k,t) = \sum_{j=1}^{n} M_t^Q((K_{T_{j-1}} - k)\frac{e^{rt}}{e^{rT_j}}).$$

As with a swap, the fixed payment, k can be chosen so that at initiation, T_0, $e(k, T_0) = 0$. The value of k for which the arrangement is valueless at time $t \leq T_0$ is

$$k_t = (\frac{K_t}{e^{rt}})\frac{\sum_{j=1}^{n} e^{-r(T_j - T_{j-1})}}{\sum_{j=1}^{n} e^{-rT_j}}.$$

We note that this means the process (k_t) is a *martingale* on $[0, T_0]$. If we assume that the time interval between dates is a constant, δ, say, then

$$k_t = (\frac{K_t}{e^{rt}})\, n\, e^{-rT_0} (\frac{e^{rT_n} - e^{rT_{n-1}}}{e^{rT_n} - e^{rT_0}}).$$

So in this case the martingale (k_t) is given by

$$k_t = \left(e^{\sigma W_t - \frac{\sigma^2}{2} t}\right) n\, e^{rT_0} (\frac{e^{rT_n} - e^{rT_{n-1}}}{e^{rT_n} - e^{rT_0}}).$$

We use this directly.

1.2 An option on the Arrangement

Suppose now that at time $0 < T_0$ we buy an option to enter into the exchange arrangement at time T_0 with the fixed payment set at k. We imagine also that we can at time T_0 enter into the arrangement with the fixed payment set at the "market" level of k_{T_0}. Since $e(k_{T_0}, T_0) = 0$ the value at time T_0 of the option will be

$$e(k, T_0) - e(k_{T_0}, T_0) = \sum_{j=1}^{n} M_{T_0}^{Q} (k_{T_0} - k) \frac{e^{rT_0}}{e^{rT_j}}.$$

So this is positive so long as $k_{T_0} > k$ which identifies the exercise region for this option. Therefore, the value of the option at time T_0 is

$$O(e(k, T_0)) = \sum_{j=1}^{n} M_{T_0}^{Q} (k_{T_0} - k)^{+} \frac{e^{rT_0}}{e^{rT_j}}.$$

The value at time $t < T_0$ will be

$$O(e(k, t)) = e^{r(t - T_0)} M_t^Q (\sum_{j=1}^{n} M_{T_0}^{Q} (k_{T_0} - k)^{+} \frac{e^{rT_0}}{e^{rT_j}}).$$

1.3 The option price at time $t = 0$

We assume that $T_i - T_{i-1} = \delta$. The price is

$$E^Q((k_{T_0} - k) I_{\{k_{T_0} > k\}} \sum_{j=1}^{n} \frac{1}{e^{rT_j}}).$$

Noting that

$$k_{T_0} \sum_{j=1}^{n} \frac{1}{e^{rT_j}} = K_{T_0} e^{-rT_0}\, n\, e^{-r\delta}$$

that

$$k_{T_0} > k \iff K_{T_0} > \frac{k}{n\left(\frac{e^{rT_n} - e^{rT_{n-1}}}{e^{rT_n} - e^{rT_0}}\right)} = \kappa_n$$

Therefore

$$E^Q(k_{T_0} I_{\{k_{T_0} > k\}} \sum_{j=1}^{n} \frac{1}{e^{rT_j}}) = e^{-rT_0}\, n\, e^{-r\delta} E^Q(K_{T_0} I_{\{K_{T_0} > \kappa_n\}}).$$

The last term on the right is familiar. Since $\sum_{j=1}^{n} \frac{1}{e^{rT_j}} = \frac{n\, e^{-r\delta} e^{-rT_0} \kappa_n}{k}$ the other term is

$$e^{-rT_0} n\, e^{-r\delta}\, \kappa_n\, Q(\{K_{T_0} > \kappa_n\}).$$

And this is very familiar too. The price appears now as

$$n\, e^{-r\delta}(K_0 N(d_1) - e^{-rT_0}\, \kappa_n N(d_2))$$

That is, a multiple of the Black-Scholes price of an option stuck on K with strike κ_n and expiry T_0.

2 The Option on the arrangement with Barriers

The option begins at, say, time $t = 0$. The underlying K s monitored continuously over the time period $[0, T_0]$. If $\max_{[0,T_0]} K_t \geq H$ then the option ends. Otherwise, the exchange arrangement begins. But all of this is subject to the condition that $\max_{[0,T_{i-1}]} K_t < H$. If, subsequently, $\max_{[T_{i-1},T_i]} K_t \geq H$ the arrangement ceases at time T_i when (final) exchange payments are made. So this arrangement can fail to begin at time T_0, last for only one time period beyond T_0, or two time periods, and so on. Because of the protocol adopted for termination, should the exchange arrangement be unterminated at time T_{n-1} then it continues to its conclusion at time T_n irrespective of what K does in the time interval $(T_{n-1}, T_n]$. For this reason the option is described as a *tenor varying* exchange arrangement. It is similar to a barrier swap, but the barrier remains in place after the arrangement has begun. Also, the underlying, K, which we have assumed to be log-normal, is given at the outset and is not derived from or related to a model of bond prices. We have assumed the existence of a cash account process (e^{rt}) which along with K lives on the base $(\Omega, \mathcal{F}_{T_n}, \mathcal{F}_t, [0, T_n], Q)$, and Q is the risk-neutral measure.

2.1 The payoff of the option on the exchange arrangement.

To describe the payoff of the option we introduce the first time that K hits the barrier H;

$$\tau(w) = \min[t : \; K_t \;\geq\; H].$$

This is a stopping time of the filtration. If $0 \leq \tau \leq T_0$ then the payoff is zero. If, for $1 \leq i \leq n-1$, $T_{i-1} < \tau \leq T_i$ then the arrangement ends at time T_i and the payoff at time T_n for the party paying the fixed amount k will be

$$\sum_{j=1}^{i} (k_{T_0} - k)^+ e^{r(T_n - T_j)}.$$

If $\tau > T_{n-1}$ then the (full) payoff (at time T_n) occurs,

$$\sum_{j=1}^{n} (k_{T_0} - k)^+ e^{r(T_n - T_j)}.$$

The payoff, at time T_n, for the party paying the fixed amount k is therefore

$$e(T_0, T_n, k) = \sum_{i=1}^{n-1} (\sum_{j=1}^{i} (k_{T_0} - k)^+ e^{r(T_n - T_j)}) I_{(T_{i-1} < \tau \leq T_i]} + \sum_{j=1}^{n} (k_{T_0} - k)^+ e^{r(T_n - T_j)} I_{(T_{n-1} < \tau \leq \infty]}.$$

The value of this payoff at time $t < T_0$ is therefore

$$\begin{aligned} e^{rt} M_t^Q(e(T_0, T_n, k) e^{-rT_n}) &= e^{rt} \sum_{i=1}^{n-1} (\sum_{j=1}^{i} M_t^Q((k_{T_0} - k)^+ e^{-rT_j} I_{(T_{i-1} < \tau \leq T_i]})) \\ &+ e^{rt} \sum_{j=1}^{n} M_t^Q((k_{T_0} - k)^+ e^{-rT_j} I_{(T_{n-1} < \tau \leq \infty]}). \end{aligned}$$

2.2 The price at time zero

For the value at time $t = 0$ we replace M^Q with E^Q in the last equation above. Let us consider a typical term and simplify by assuming $T_i - T_{i-1} = \delta$. We have

$$\sum_{j=1}^{i} E^Q((k_{T_0} - k)^+ e^{-rT_i} I_{(T_{i-1} < \tau \leq T_i]}).$$

We also have

$$k_{T_0} = K_{T_0} n \frac{e^{rT_n} - e^{rT_{n-1}}}{e^{rT_n} - e^{rT_0}}.$$

We have already observed that $\{k_{T_0} > k\} = \{K_{T_0} > \kappa_n\}$ where

$$\kappa_n = \frac{k}{n} \frac{e^{rT_n} - e^{rT_0}}{e^{rT_n} - e^{rT_{n-1}}}$$

So

$$(k_{T_0} - k) I_{\{k_{T_0} > k\}} = k\, (\frac{K_{T_0}}{\kappa_n} - 1) I_{\{\frac{K_{T_0}}{\kappa_n} > 1\}}$$

So our typical term contributes

$$E^Q(k\, (\frac{K_{T_0}}{\kappa_n} - 1) I_{\{\frac{K_{T_0}}{\kappa_n} > 1\}} \sum_{j=1}^{i} e^{-rT_i} I_{(T_{i-1} < \tau \leq T_i]}).$$

Now we know already what $\sum_{j=1}^{i} e^{-rT_j}$ amounts to. By analogy with the definition of the constant κ_n we define

$$\kappa_i = e^{rT_1} \frac{k}{i} \sum_{j=1}^{i} e^{-rT_j}$$

This allows a moderate simplification to the term we are interested in, it now looks like

$$e^{-rT_1} \kappa_i E^Q((\frac{K_{T_0}}{\kappa_n} - 1) I_{\{\frac{K_{T_0}}{\kappa_n} > 1\}} I_{(T_{i-1} < \tau \leq T_i]}).$$

Writing $\hat{K_t} = \frac{K_t}{\kappa_n}$, $\hat{H} = \frac{H}{\kappa_n}$ and noting that τ is identical with the first time that $\hat{K}$ is greater or equal to $\hat{H}$ we see that the quantity we have to calculate is

$$e^{-rT_1}\kappa_i E^Q((\hat{K}_{T_0} - 1)I_{\{\hat{K}_{T_0}>1\}}I_{\{M^{\hat{K}}_{T_{i-1}}<\hat{H}\ ,\ M^{\hat{K}}_{T_i}\geq\hat{H}\}}).$$

So we can forget the "^'s" we just have remember that $\hat{K}_0 = \frac{K_0}{\kappa_n}$ and $\hat{H} = \frac{H}{\kappa_n}$. So now we consider

$$e^{-rT_1}\kappa_i E^Q((K_{T_0} - 1)I_{\{K_{T_0}>1,\ M^{K_{T_{i-1}}}<H\ ,\ M^{K_{T_i}}\geq H\}}).$$

It seems natural to deal first with

$$Q(\{K_{T_0} > 1,\ M^{K_{T_{i-1}}} < H\ ,\ M^{K_{T_i}} \geq H\})$$

So, in this case K lies under the level H over $[0, T_0]$ taking a value at time T_0 which is greater than 1. It remains under H in $[T_0, T_{i-1}]$ taking *any* value less than H at T_{i-1}, then, during the interval $(T_{i-1}, T_i]$ it hits the level H taking *any* value at time T_i. These three time periods form the basis of our analysis. Let $p(T_0, K_0, H, x)$ denote the probability that K starts at $t = 0$ with the value K_0, remains under the level H until T_0 and takes a value x at T_0. Let $p(T_{i-1}, x, H, y)$ be the probability that starting from a value x, K will remain under the level H over $[T_0, T_{i-1}]$ taking any value y less than H at time T_{i-1}. Let $p^{'}(T_i, y, H)$ be the probability that K takes the value $y < H$ at time T_{i-1} and its running maximum over $(T_{i-1}, T_i]$ exceeds H. These probabilities are independent (...reborn Brownian Motion...)and the probability we wish to calculate is given by a formal integral

$$\int_{-\infty}^{H}\int_{1}^{H} p(T_0, K_0, H, x)p(T_{i-1}, x, H, y)p^{'}(T_i, y, H)dx\,dy \quad (1)$$

We recall that

$$log(\frac{K_t}{K_0}) = \sigma(W_t + (\frac{r}{\sigma} - \frac{\sigma}{2})t)$$

so we will deal with the drifted Brownian Motion. In terms of this the calculation has the same form as in (1) and it is clear that the joint distribution of a drifted Brownian motion and its running maximum will be useful.We note that

$$\begin{aligned} 1 < K_{T_0} < H \quad &\Longleftrightarrow \quad a_0 < B^{\nu}_{T_0} < b_0 \\ M^K_{T_0} < H \quad &\Longleftrightarrow \quad M^{B^{\nu}}_{T_0} < b_0 \end{aligned}$$

where $a_0 = \frac{1}{\sigma}\log(\frac{1}{K_0})$, $b_0 = \frac{1}{\sigma}\log(\frac{H}{K_0})$ and B^{ν}_t is the drifted Brownian Motion

$$B^{\nu}_t = W_t + (\frac{r}{\sigma} - \frac{\sigma}{2})t \text{ and } \nu = \frac{\text{r}}{\sigma} - \frac{\sigma}{2}.$$

The joint distribution of B^{ν} and $M^{B^{\nu}}$ with respect to the measure Q is well known and we shall be using a partial derivative of this distribution,

$$Q(\{M^{B^{\nu}}_{T_0} < b_0, B^{\nu}_{T_0} = x\}) = \frac{1}{\sqrt{T_0}}\left(\phi(\frac{x - \nu T_0}{\sqrt{T_0}}) - e^{2b_0\nu}\phi(\frac{x - 2b_0 - \nu T_0}{\sqrt{T_0}})\right)dx$$

which gives the probability on the left above for a drifted Brownian Motion *starting at* 0 *at time* 0. For the period $[T_0, T_{i-1}]$ where we need to consider B^ν starting at time T_0 at the value x and staying below the level b_0 throughout $[T_0, T_{i-1}]$ and taking a value y at time T_{i-1} we can obtain the probabilty by 'translating' everything by the value x, this relys on our Brownian motion being 'reborn' at time T_0. Without giving the full details of the argument; the probability we require is the same as the probability that the drifted Brownian Motion, B^ν starts from zero and over a time period of length $T_{i-1} - T_0$ never reaching the value $b_0 - x$ and takeing a value $y - x$ at time T_{i-1}. According to our distirbution above the probability

$$Q(\{M^{B^\nu}_{T_{i-1}-T_0} < b_0 - x, B^\nu_{T_{i-1}-T_0} = y - x\})$$

will be

$$\frac{1}{\sqrt{T_{i-1} - T_0}} \left(\phi(\frac{y - x - \nu(T_{i-1} - T_0)}{\sqrt{T_{i-1} - T_0}}) - \right.$$
$$\left. e^{2(b_0 - x)\nu} \phi(\frac{(y - x) - 2(b_0 - x) - \nu(T_{i-1} - T_0)}{\sqrt{T_{i-1} - T_0}}) \right) dy.$$

For the third time period we require the probability that starting from a value y our drifted Brownian Motion will exceed the level b_0 during the time period $(T_{i-1}, T_i]$ and much as before, we calculate $Q\{M^{B^\nu}_{T_i - T_{i-1}} \geq b_0 - y\}$. We can read this off from the joint distribution of B^ν and its running maximum;

$$Q(\{M^{B^\nu}_{T_i - T_{i-1}} < b_0 - y, B^\nu_{T_i - T_{i-1}} < z\})$$

is equal to

$$N(\frac{z - \nu(T_i - T_{i-1})}{\sqrt{T_i - T_{i-1}}}) - e^{2(b_0 - y)\nu} N(\frac{z - 2b_0 - \nu(T_i - T_{i-1})}{\sqrt{T_i - T_{i-1}}}).$$

But of course $z \leq b_0 - y$. Letting $z = b_0 - y$ leaves us with

$$N(\frac{b_0 - y - \nu(T_i - T_{i-1})}{\sqrt{T_i - T_{i-1}}}) - e^{2(b_0 - y)\nu} N(\frac{-y - b_0 - \nu(T_i - T_{i-1})}{\sqrt{T_i - T_{i-1}}}).$$

Of course what we want is one minus this probability:

$$N(\frac{y + \nu(T_i - T_{i-1}) - b_0}{\sqrt{T_i - T_{i-1}}}) + e^{2(b_0 - y)\nu} N(\frac{-y - b_0 - \nu(T_i - T_{i-1})}{\sqrt{T_i - T_{i-1}}}).$$

To calculate the probability, (1);

$$Q(\{K_{T_0} > 1,\ M^{K_{T_{i-1}}} < H\ ,\ M^{K_{T_i}} \geq H\})$$

we must integrate the product of the three probabilities determined above over the appropriate limits, a_0 to b_0 for the x variable and $-\infty$ to b_0 for the y variable. After this these must be summed from $i = 1$ to $i = n - 1$.

This deals with the terms

$$Q(\{K_{T_0} > 1,\ M^{K_{T_{i-1}}} < H\ ,\ M^{K_{T_i}} \geq H\})$$

and there remains

$$e^{-rT_1}\kappa_i E^Q(K_{T_0} I_{\{K_{T_0}>1,\ M^{K_{T_{i-1}}}<H\ ,\ M^{K_{T_i}}\geq H\}}).$$

We can use the familiar change of measure technique; $K_0 e^{\sigma B_t+(r-\frac{\sigma^2}{2})t}$ and $e^{\sigma B_t+(r-\frac{\sigma^2}{2})t}$ provides the Radon-Nikodym derivative for a change of measure to, say, $Q^{'}$, under which there is a change of sign in the dynamics for K. This leads to minor changes in the the calculations we have already outlined.

We have one last term to consider: At time $t < T_0$ the event $\tau > T_{n-1}$ contributes

$$e^{rt}\sum_{j=1}^{n} M_t^Q((k_{T_0}-k)^+ e^{-rT_j} I_{(T_{n-1}<\tau\leq\infty]})$$

which amounts to

$$\sum_{j=1}^{n} E^Q((k_{T_0}-k)^+ e^{-rT_j} I_{(T_{n-1}<\tau\leq\infty]})$$

at time 0. Using the analysis and the notations adopted for the previous case this is equivalent to

$$e^{-rT_1}\kappa_n E^Q((K_{T_0}-1) I_{\{K_{T_0}>1,\ M^{K_{T_{n-1}}}<H\}},$$

since $M^{K_{T_{n-1}}} < H \iff \tau > T_{n-1}$. But we have already discussed how to evalute an expectation like this. In terms of the formal integral described in equation (1) above the probability we will need to calculate is

$$\int_{-\infty}^{H}\int_{1}^{H} p(T_0, K_0, H, x) p(T_{n-1}, x, H, y) dx\, dy \quad (2).$$

As before we split this into two parts. We calculate the probability as before and then move to the term involving K_{T_0}.This is dealt with by the change of measure technique alluded to above.

As one can imagine, the complete calculation generates a large number of terms involving the integration of products of trios of Normal densities and Normal distributions. Partial integration of some of these terms produces some simplification but increases the number of terms. Numerically, the task is quite staightforward and the computation time with Maple is of the order of seconds.

2.3 A Standard Barrier Arrangement

The reader will have noted that we have not looked at the case of a 'standard' barrier option on the arrangement where the underlying is monitored over $[0, T_0]$ only and should it reach the level H during this monitoring period the option expires worthless. It is reasonable that such a barrier option will be dearer than the option on tenor varying arrangement. Indeed a simple formal calculation confirms this. As one might expect, the price of the tenor varying option is an increasing function of H, the barrier level, and that the price of both the standard and tenor varying options converge to that of the option without barriers as H increases to infinity. Again, as one expects, the price is an increasing function of the length of the arrangement.

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2.4 Approximation of the Price

The computations were achieved using Maple. The first graph shows some plots of price against barrier level. The time periods are of equal length in this calculation and three different 'tenors' are calculated. The output is in terms of basis points. The reader will see "C.T.B.S." and 'Swap' in the legend. This and our other diagrams arise from [1] where the "C" refers to continuous monitoring of the underlying the exchange arrangement is described as a swap. Of course this courts confusion with genuine Swaps and the term is deprecated here. However there is a paper in preparation which usess a Swap Market Model for the underlying k. The discretely monitored case of the tenor varying arrangement was discussed in [1] and 'barrier shifting' used to approximate the price in this case.

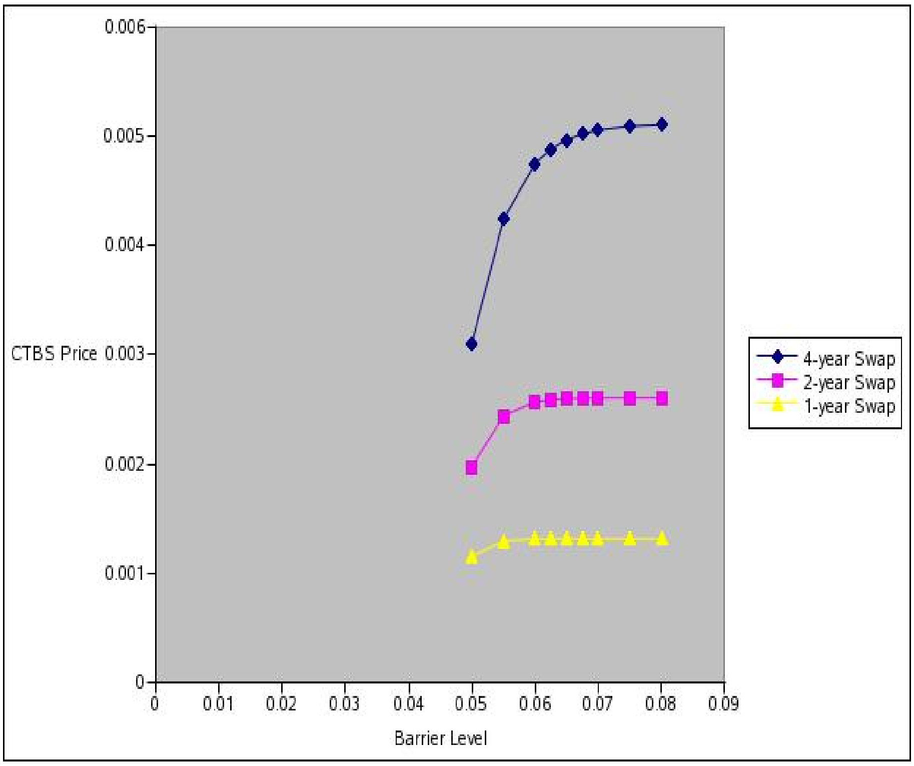

Figure 1: We consider the case where risk free rate r is at 2 percent, the standard deviation σ is at 10 percent, k_0 is equal to 4 percent, and the contract starts (at T_0) six months later and interest is paid quarterly. The strike rate is equal to k_0.

Using the same data set as above the price of the option on the exchange arrangement is equal to 51.10477 basis points. We see that the price of the tenor varying option is significantly less.

3 References

[1] Tanapan Tanpradist. Discretely Monitored Tenor Varying Exchange Arrangements. PhD Thesis. Imperial College 2005.
[2] Tanapan Tanpradist. A Swap Market Model with a Tenor Varying Swap. Preprint December 2007.