

is an open set in  $\mathbb{R}$  then  $F^{-1}(E)$  is an open set in  $\mathbb{R}^d$ .

The collection  $M = \{H \subseteq \mathbb{R} : F^{-1}(H) \text{ is a Borel set in } \mathbb{R}^d\}$  is a  $\sigma$ -field because inverse images preserve set operations:

$$F^{-1}(\mathbb{R} \setminus H) = F^{-1}(\mathbb{R}) \setminus F^{-1}(H),$$

$$F^{-1}\left(\bigcup_n H_n\right) = \bigcup_n F^{-1}(H_n).$$

Now the Borel  $\sigma$ -field is generated by the open sets of the space -  $\mathbb{R}$  or  $\mathbb{R}^d$ . As  $F^{-1}(E)$  is open whenever  $E$  is open the  $\sigma$ -field  $M$  contains the open sets of  $\mathbb{R}$  and therefore contains also the Borel  $\sigma$ -field. So a continuous function  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  is such that  $f^{-1}(F)$  is a Borel set in  $\mathbb{R}^d$  whenever  $F$  is a Borel set in  $\mathbb{R}$  ( $f$  is measurable).

As a consequence  $F(\Delta B): \Omega \rightarrow \mathbb{R}$  is 'measurable' because if  $F$  is a Borel set in  $\mathbb{R}$ ,  $F(\Delta B)^{-1}(F) = \Delta B^{-1}(F^{-1}(F))$  and  $\Delta B$  is a measurable function:  $\Omega \rightarrow \mathbb{R}$ . Ok

now take  $f(x_1, \dots, x_d) = x_i x_j$ . This is continuous.

So  $F(\Delta B) = \Delta B^i \Delta B^j$  is measurable ( $\Omega \rightarrow \mathbb{R}$ ). But we know that  $\Delta B^{-1}(H)$  is independent of  $\mathcal{F}_s$  for every Borel set  $H$  in  $\mathbb{R}^d$ . In particular  $\Delta B^{-1}(F^{-1}(F))$  is independent of  $\mathcal{F}_s$  for every Borel set  $F$  in  $\mathbb{R}$ . So  $F(\Delta B) = \Delta B^i \Delta B^j$  is independent of  $\mathcal{F}_s$ . Consequently

$$M_s^{\mathbb{P}}(\Delta B^i \Delta B^j) = \mathbb{E}(\Delta B^i \Delta B^j) \mathbb{I}_{\Omega} \quad \Delta B^i \Delta B^j \text{ indep of } \mathcal{F}_s$$

$$= \mathbb{E}(\Delta B^i) \mathbb{E}(\Delta B^j) \mathbb{I}_{\Omega} \quad \Delta B^i \text{ indep of } \Delta B^j$$

$$= 0$$

Now because  $B_t^i B_t^j = B_s^i B_s^j + B_s^i (B_t^j - B_s^j) + B_s^j (B_t^i - B_s^i) + (B_t^i - B_s^i)(B_t^j - B_s^j)$