example, if $H = H_1 \times H_2 \times \ldots \times H_d$ where $H_i \in \mathcal{B}(\mathbb{R})$ then

$$\mathbb{P}(\Delta B \in H) = \int_{H_d}\int_{H_{d-1}}\cdots\int_{H_2}\int_{H_1}\left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^d e^{-\frac{(x_1^2+\cdots+x_d^2)}{2(t-s)}}\,dx_1\,dx_2\cdots dx_{d-1}dx_d$$

Taking $H_\ell = \mathbb{R}$ for $\ell \neq j$ and i and noting

$$e^{-\frac{(x_1^2+\cdots+x_d^2)}{2(t-s)}} = \prod_{\ell=1}^{d} e^{-x_\ell^2/2(t-s)}$$

we can integrate out the x_ℓ variables for $\ell \neq i, j$ to leave

$$\mathbb{P}(\Delta B \in H) = \underbrace{\mathbb{P}(\Delta B^i \in H_i, \Delta B^j \in H_j)}_{\text{(and the } s\text{-fold one!)}} = \int_{H_i}\int_{H_j}\left(\frac{1}{\sqrt{2\pi(t-s)}}\right)^2 e^{-x_i^2/2(t-s)} \cdot e^{-x_j^2/2(t-s)}\,dx_j\,dx_i$$

of course this integral factorises into the product

$$\left(\frac{1}{\sqrt{2\pi(t-s)}}\int_{H_i} e^{-x_i^2/2(t-s)}\,dx_i\right)\left(\frac{1}{\sqrt{2\pi(t-s)}}\int_{H_j} e^{-x_j^2/2(t-s)}\,dx_j\right).$$

Two things follow: Setting $H_j = \mathbb{R}$ shows that for each i, $B_t^i - B_s^i$ is normally distributed with mean 0 and variance $t-s$. Since ΔB is independent of $\mathcal{F}_s$ then the event $\{\Delta B \in H\}$ is independent of $\mathcal{F}_s$, and in our special case this tells us that $B_t^i - B_s^i$ is independent of $\mathcal{F}_s$. So B^i is a Brownian Motion. Also the expression for $\mathbb{P}(\Delta B^i \in H_i, \Delta B^j \in H_j)$ is equal to $\mathbb{P}(\Delta B^i \in H_i)\mathbb{P}(\Delta B^j \in H_j)$ that is, ΔB^i is independent of ΔB^j. Since $B_0^i = 0$ $\forall i$, $\mathbb{P}$a.s. then B_t^i and B_t^j are independent for every $t \in \mathbb{R}$. There's more! First of all let $f : \mathbb{R}^d \to \mathbb{R}$ be a <u>continuous function</u>; $\forall \underline{x}, \forall \epsilon > 0, \exists \delta > 0 : \|\underline{x} - \underline{y}\|_d < \delta$ $|f(\underline{x}) - f(\underline{y})| < \epsilon$. A consequence of this is that if E