0·1 Multidimensional Brownian Motion

A d-dimensional Brownian Motion $B = (B_1, \dots, B_d)$, with initial distribution μ is a continuous adapted process with values in $\mathbb{R}^d$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying

(i) $\mathbb{P}(B_0 \in H) = \mu(H) \quad \forall H \in \mathcal{B}(\mathbb{R}^d)$

(ii) For $0 \leqslant s < t$, $B_t - B_s$ is independent of $\mathcal{F}_s$ and is normally distributed(†) with mean zero and covariance matrix $(t-s)I_d$, here I_d is the $d \times d$ identity matrix.

Usually we take μ so $\mathbb{P}(B_0 = 0) = 1$, i.e. $\mu\{0\} = 1$.

(†) X Normally distributed with mean $\underline{\mu} = (\mu_1, \dots, \mu_d)$ and covariance matrix $V = [v_{ij}]$ means: for $H \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathbb{P}(X \in H) = \int_{\mathbb{R}^d} I_H \frac{1}{\sqrt{(2\pi)^d |V|}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu}) V^{-1} (\underline{x}-\underline{\mu})'} d\underline{x}$$

and $|V| = \det V$

So

$$\mathbb{P}(\Delta B \in H) = \int_{\mathbb{R}^d} I_H \cdot \frac{1}{\sqrt{(2\pi)^d (t-s)^d}} e^{-\frac{1}{2}(\underline{x}\cdot\underline{x}')/(t-s)} d\underline{x}$$

$$= \int_{\mathbb{R}^d} I_H \cdot \frac{1}{\sqrt{(2\pi)^d (t-s)^d}} e^{-\frac{|\underline{x}|^2}{2(t-s)}} d\underline{x} \qquad (\Delta B = B_t - B_s)$$

This integral over $\mathbb{R}^d$ can be expressed as a d-fold repeated integral when H has a simple form: for