Some abstract Measure and Integration Theory (background for Probability)

σ-field (sigma field or sigma algebra)

Let Ω be any given non-empty set. For each subset A of Ω define its complement A^c by $\Omega - A$, (where $B - A = \{w : w \in B \text{ and } w \notin A\}$).

A σ-field $\mathcal{F}$ on Ω is a collection of subsets of Ω such that

(1) ϕ and $\Omega \in \mathcal{F}$;

(2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$;

(3) if $\{A_n\}$ is any countable (i.e. finite or denumerable) collection of sets in $\mathcal{F}$ then $\bigcup_n A_n \in \mathcal{F}$.

It follows from (1), (2) and (3) that if $\{A_n\}$ is any countable collection of sets in $\mathcal{F}$ then $\bigcap_n A_n \in \mathcal{F}$ also, (using the fact that $(\bigcap_n A_n)^c = \bigcup_n A_n^c$).

If $\mathcal{F}'$ is another σ-field on Ω and $\mathcal{F}' \subseteq \mathcal{F}$ then $\mathcal{F}'$ is said to be smaller than $\mathcal{F}$, $\mathcal{F}$ larger than $\mathcal{F}'$.

Ex.1 If Ω has exactly n elements and $\mathcal{F}$ is the collection of all subsets of Ω , then $\mathcal{F}$ is a σ-field and consists of 2^n sets.

Ex.2 A and $B \in$ σ-field $\mathcal{F} \Longrightarrow A - B \in \mathcal{F}$.

Ex.3 If Ω is any non-empty set and $\mathcal{F}$ is the collection of all subsets of Ω , then $\mathcal{F}$ is the largest σ-field on Ω ; the smallest σ-field on Ω consists of just ϕ and Ω .

Ex.4 Let ξ be any collection of subsets of a given non-empty set Ω . Then there exists a smallest σ-field on Ω which contains ξ , namely the intersection of all σ-fields on Ω which contain ξ .

Measure

A measure μ is a countably additive non-negative set function defined on the sets in a σ-field $\mathcal{F}$; this means that

(4) $0 \le \mu(A) \le \infty \quad \forall A \in \mathcal{F}$, ($\mu(A)$ is called the measure of A) ;

(5) $\mu(\phi) = 0$;

(6) if $\{A_n\}$ is any countable collection of sets in $\mathcal{F}$ which are pairwise disjoint (i.e. $A_n \cap A_m = \phi$ for $n \neq m$) then

$$\mu(\bigcup_n A_n) = \sum_n \mu(A_n) \quad .$$

Ex.5 If $A, B \in \mathcal{F}$ and $A \subseteq B$ then $\mu(A) \leq \mu(B)$.

For any countable collection $\{A_n\}$ of sets in $\mathcal{F}$

(7) $$\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n) \quad ,$$

[for if one puts $B_1 = A_1$, $B_2 = A_2 - A_1$, and generally $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$ for $n \geq 2$, then $\bigcup_n A_n = \bigcup_n B_n$ with the B_n's pairwise disjoint, so that $\mu(\bigcup_n A_n) = \mu(\bigcup_n B_n) = \sum_n \mu(B_n) \leq \sum_n \mu(A_n)$ using (6) and Ex.5] .

The triple $(\Omega, \mathcal{F}, \mu)$ is called a measure space and the sets in $\mathcal{F}$ are called measurable. If $\mu(\Omega) = 1$, μ is called a probability measure and usually denoted by P (or pr) whilst the triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

Continuity Properties of Measure Suppose that $\{A_n\}$ is any sequence of sets in $\mathcal{F}$.

(8) If $A_n \subseteq A_{n+1}$ for $n \geq 1$ (written $A_n \nearrow$) then $\mu(A_n) \nearrow \mu(A)$ as $n \to \infty$ where $A = \bigcup_n A_n$. [$A \in \mathcal{F}$ by (3).]

(9) If $A_n \supseteq A_{n+1}$ for $n \geq 1$ (written $A_n \searrow$) and $\mu(A_{n_1}) < \infty$ for some n_1 then $\mu(A_n) \searrow \mu(A)$ as $n \to \infty$ where $A = \bigcap_n A_n$; in particular

N.B. $\mu(A_n) \searrow 0$ when $\bigcap_n A_n = \phi$.

Almost Everywhere (a.e.) If a certain property is true for all elements w in a set $E \in \mathcal{F}$ s.t. $\mu(E^c) = 0$, then that property is said to be true almost everywhere (w.r.t. μ) .

Two Important Cases:-

(a) Lebesgue Measure on $\mathbb{R}$. Let $\Omega = \mathbb{R} = (-\infty, \infty)$. It can be shown that there is a σ-field consisting of certain subsets of $\mathbb{R}$ and containing all intervals in $\mathbb{R}$ (including infinite and semi-infinite intervals), and there is a measure μ (Lebesgue measure) defined on this σ-field such that

$$\mu(I) = \text{length } I \qquad \forall \text{interval } I .$$

(b) Lebesgue Measure on $\mathbb{R}^n$. Let $\Omega = \mathbb{R}^n$. It can be shown that there is a σ-field consisting of certain subsets of $\mathbb{R}^n$ and containing all intervals in $\mathbb{R}^n$ (i.e. all sets of the form $J = I_1 \times I_2 \times \ldots \times I_n$ where $I_1, \ldots, I_n$ are intervals in $\mathbb{R}$) and there is a measure μ (Lebesgue measure) defined on this σ-field such that

$$\mu(J) = \text{"volume" of } J \qquad \forall \text{interval } J \text{ in } \mathbb{R}^n .$$

In cases (a) and (b) the sets in the σ-fields are said to be Lebesgue measurable and μ is Lebesgue measure.

Lebesgue measure is constructed with the additional property of "completeness", i.e. whenever A is Lebesgue measurable with $\mu(A) = 0$, then N is Lebesgue measurable with $\mu(N) = 0 \quad \forall N \subseteq A$. One often harmlessly assumes completeness for other measure spaces.

Measurable Functions

Suppose that $(\Omega, \mathcal{F}, \mu)$ is a given measure space and that f is a function from Ω to $\mathbb{R}$, or to $\overline{\mathbb{R}}$ $(= \mathbb{R} \cup \{-\infty, +\infty\})$.

f is said to be measurable if the set $\{w \in \Omega : f(w) < y\}$ is measurable i.e. $\in \mathcal{F} \ \forall y \in \mathbb{R}$.

It can be shown that f is measurable if and only if

$$\{w \in \Omega : f(w) \in I\} \in \mathcal{F} \quad \forall \text{interval } I ;$$

(here when f maps to $\overline{\mathbb{R}}$, one allows infinite and semi-infinite intervals possibly to include both or the appropriate one of $+\infty$ and $-\infty$).

Construction of the Integral (in stages)

Suppose that $(\Omega, \mathcal{F}, \mu)$ is given.

(i) The characteristic function or indicator function of any set $E \subseteq \Omega$ is defined by

$$\chi_E(w) = \begin{cases} 1 & \text{for } w \in E , \\ 0 & \text{otherwise} . \end{cases}$$

When $A \in \mathcal{F}$, χ_A is a measurable function and $\int_\Omega \chi_A(w)\, d\mu(w)$ is defined by $\mu(A)$

(ii) A <u>simple function</u> is a linear combination of indicator functions

$$f(w) = \alpha_1 \chi_{A_1}(w) + \ldots + \alpha_n \chi_{A_n}(w) \quad , \quad w \in \Omega \quad ,$$

where $\alpha_1, \ldots, \alpha_n$ are real constants and $A_1, \ldots, A_n \in \mathcal{F}$. Then define

$\int_\Omega f(w) d\mu(w)$ by $\sum_{i=1}^{n} \alpha_i \mu(A_i)$.

(iii) Now suppose that f is measurable and $f(w) \geq 0 \ \forall w \in \Omega$. Then it can be shown that (a) $\exists$ an increasing sequence $f_n(w)$ of non-negative simple functions such that $f_n(w) \nearrow f(w)$ as $n \to \infty \ \forall w \in \Omega$, and $\lim_{n\to\infty} \int_\Omega f_n(w) d\mu(w)$ exists (possibly $= +\infty$) , and (b) one may define $\int_\Omega f(w) d\mu(w)$ uniquely by this limit (i.e. irrespective of any particular choice of sequence $f_n(w)$).

(iv) Next suppose that f is measurable but not necessarily non-negative. Then define non-negative functions f^+ and f^- such that $f(w) = f^+(w) - f^-(w)$ $\forall w \in \Omega$, by putting $f^+(w) = f(w)$ and $f^-(w) = 0$ when $f(w) \geq 0$

and $f^+(w) = 0$ and $f^-(w) = -f(w)$ when $f(w) < 0$.

Then using (iii) on f^+ and f^- , define

$\int_\Omega f(w) d\mu(w)$ by $\int_\Omega f^+(w) d\mu(w) - \int_\Omega f^-(w) d\mu(w)$ when the last two integrals are not both $+\infty$; otherwise $\int_\Omega f(w) d\mu(w)$ is not defined.

If $A \in \mathcal{F}$ define $\int_A f(w) d\mu(w)$ by $\int_\Omega \chi_A(w) f(w) d\mu(w)$ when the latter integral exists.

Often abbreviate $\int_\Omega f(w) d\mu(w)$ to $\int_\Omega f d\mu$ or $\int f d\mu$, and $\int_A f(w) d\mu(w)$ to $\int_A f d\mu$.

If $\int_A f d\mu$ is finite, f is said to be <u>integrable over A w.r.t. μ</u> , or just integrable w.r.t. μ when $A = \Omega$.

<u>Properties of the Integral.</u> Suppose that $(\Omega, \mathcal{F}, \mu)$ is a given measure space.

I Suppose that A and B are any disjoint sets in $\mathcal{F}$, and f and g are integrable w.r.t. μ .

(i) αf (α = real constant), $f+g$ and $|f|$ are integrable; f is integrable over A.

(ii) $\int_{A \cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu$.

(iii) f is finite a.e.

(iv) $\int_A (f+g)\,d\mu = \int_A f\,d\mu + \int_A g\,d\mu$; $\int_A (\alpha f)\,d\mu = \alpha \int_A f\,d\mu$.

(v) $\left|\int_A f\,d\mu\right| \le \int_A |f|\,d\mu$.

(vi) $f \ge 0$ a.e. $\Rightarrow \int_\Omega f\,d\mu \ge 0$; $f \ge g$ a.e. $\Rightarrow \int_\Omega f\,d\mu \ge \int_\Omega g\,d\mu$.

(vii) $f \ge 0$ a.e. and $\int_\Omega f\,d\mu = 0 \Rightarrow f = 0$ a.e.

(viii) $f = g$ a.e. $\Rightarrow \int_\Omega f\,d\mu = \int_\Omega g\,d\mu$.

(ix) h measurable and $|h| \le f$ a.e. $\Rightarrow h$ is integrable.

II <u>Absolute Continuity</u>. If f is integrable then $\int_A f\,d\mu \to 0$ as $\mu(A) \to 0$ with $A \in \mathcal{F}$, (i.e. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\left|\int_A f\,d\mu\right| < \varepsilon$ whenever $\mu(A) < \delta$ with $A \in \mathcal{F}$) .

III <u>Montone Convergence Theorem</u>. If $f_n \ge 0$ and f_n is measurable $\forall n$, and if $f_n(w) \nearrow f(w)$ a.e. then f is measurable and $\int f_n\,d\mu \nearrow \int f\,d\mu$.

IV <u>Fatou's Lemma</u>. Suppose that f_n is measurable $\forall n$.

(i) $f_n \le g$ a.e. with g integrable $\Rightarrow \limsup_{n\to\infty} \int f_n\,d\mu \le \int (\limsup_{n\to\infty} f_n)\,d\mu$.

(ii) $f_n \ge h$ a.e. with h integrable $\Rightarrow \liminf_{n\to\infty} \int f_n\,d\mu \ge \int (\liminf_{n\to\infty} f_n)\,d\mu$.

V <u>Dominated Convergence Theorem</u>. Suppose that f_n is measurable $\forall n$, $|f_n| \le g$ a.e. with g integrable, and $f_n \to f$ a.e. Then f is integrable and $\int f_n\,d\mu \to \int f\,d\mu$.

VI Suppose that f is Riemann integrable on a finite interval $[a,b]$, or that the improper integrals of f <u>and</u> $|f|$ converge on a possibly infinite interval. Then f is integrable w.r.t. Lebesgue measure on that interval and the two kinds of integral of f are equal.