§9 The Riemann Integral

Recall the definition of Riemann integrability for a bounded $f : \mathbb{R} \to \mathbb{R}$

Let $a \leqslant b$, a partition or dissection of $[a,b]$ is a finite subset of $[a,b]$ containing a and b. $\mathcal{P}[a,b]$ denotes the set of all partitions of $[a,b]$. Given $P \in \mathcal{P}[a,b]$, $P = \{a = x_0 < x_1 < \dots < x_n\}$

$\Delta x_i = x_i - x_{i-1}$ $\quad 1 \leqslant i \leqslant n$ and $m_i(f) = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$,

$M_i(f) = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$. Then if $U(P,f) = \sum_i M_i(f) \Delta x_i$, $L(P,f) = \sum_i m_i(f)\Delta x_i$

and
$$R\int^{*} f = \inf\Big\{ \underbrace{\sum_i M_i(f)\Delta x_i}_{=U(P,f)} : P \in \mathcal{P}[a,b] \Big\}$$

$$R\int_{*} f = \sup\Big\{ \underbrace{\sum_i m_i(f)\Delta x_i}_{=L(P,f)} \cdot P \in \mathcal{P}[a,b] \Big\}$$

We say f is Riemann integrable over $[a,b]$ if $R\int^{*} f = R\int_{*} f$

and define $\displaystyle\int_a^b f = R\int^{*} f = R\int_{*} f$.

Theorem 9.1

Let f be Riemann integrable over $[a,b]$. Then $f\chi_{[a,b]}$ is Lebesgue integrable and $\displaystyle\int_a^b f = \int_{\mathbb{R}} f\chi_{[a,b]}$

Proof

A step function, g, is an upper function (obvious). Since $-g$ is also a step function then g is also a lower function. Define step functions g and h as follows.

Let $\epsilon > 0$. Choose $P \in \mathcal{P}[a,b]$ so that $U(P,f) - L(P,f) < \epsilon$

Let $$g = \sum_{i=1}^{n} M_i(f)\chi_{[x_{i-1},x_i)} + M_n(f)\chi_{\{b\}}$$

$$h = \sum_{i=1}^{n} m_i(f)\chi_{[x_{i-1},x_i)} + m_n(f)\chi_{\{b\}}$$

Then g is upper and $\int_{\mathbb{R}} g = U(P,f)$ and $g \geq f\chi_{[a,b]}$

also h is lower and $\int_{\mathbb{R}} h = L(P,f)$ and $h \leq f\chi_{[a,b]}$

So $\int_{\mathbb{R}}^{*} f\chi_{[a,b]} - \int_{*\mathbb{R}} f\chi_{[a,b]} < \epsilon$, i.e. $f\chi_{[a,b]}$ is Lebesgue integrable and since $\int_{\mathbb{R}} h \leq \int_a^b f \leq \int_{\mathbb{R}} g$ and

$\int_{\mathbb{R}} h \leq \int_{\mathbb{R}} f\chi_{[a,b]} \leq \int_{\mathbb{R}} g$ then we must have

$$\int_a^b f = \int_{\mathbb{R}} f\chi_{[a,b]} .$$

$\square$

9.2 Theorem

Suppose $|f| \leqslant M\chi_{[a,b]}$ where $M > 0$. Then f is Riemann integrable if and only if f is continuous almost everywhere.

Proof

Suppose f is continuous almost everywhere and let $N = \{x \in [a,b] : f \text{ is not continuous at } x\}$, we have $\mu N = 0$

Let $\epsilon > 0$ and (I_n) a sequence of open intervals with $N \subseteq \bigcup_n I_n$ and $\sum_n \mu I_n < \frac{\epsilon}{4M}$. If $x \in [a,b]\setminus N$ $\exists$ an open interval, centre x, I_x say, with $|f(x) - f(y)| < \epsilon/_{b-a}$ for $y \in I_x$.

Now $\{I_n : n \in \mathbb{N}\} \cup \{I_x : x \in [a,b]\setminus N\}$ is an open cover of $[a,b]$. So there is a finite subcover, say $I_{n_1}, \dots, I_{n_k}$, $I_{x_1}, \dots I_{x_\ell}$, $k, \ell \in \mathbb{N}$.

Now define step functions g, h

$$g(y) = f(x_m) - \epsilon/(b-a)\cdot 4 \quad \text{if } y \in I_{x_m} \setminus \bigcup_{r=1}^{m-1} I_r$$

$$h(y) = f(x_m) + \epsilon/(b-a)\cdot 4 \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"}$$

$$\left.\begin{array}{l} g(y) = -M \\ h(y) = M \end{array}\right\} \text{if } y \in [a,b] \setminus \bigcup_{r=1}^{\ell} I_{x_r} \subseteq \bigcup_{r=1}^{k} I_{n_r}$$

Then $g \leqslant f\chi_{[a,b]} \leqslant h$ and $\int_{\mathbb{R}} h-g = \underset{①}{\int_{[a,b] \setminus \bigcup_{r=1}^{\ell} I_{x_r}} (h-g)} +$

$\underset{②}{\int_{\bigcup_{r=1}^{\ell} I_{x_r}} (h-g)} + \underset{③}{\int_{\mathbb{R} \setminus \left(\bigcup_{r=1}^{k} I_{n_r} \cup \bigcup_{r=1}^{\ell} I_{x_r}\right)} h-g}$ (think of the finite sums involved), so, using $|\int \cdot| \leqslant \int |\cdot|$,

$$\int_{\mathbb{R}} (h-g) \leqslant \underset{②}{2 \cdot \frac{\epsilon}{4(b-a)} \cdot (b-a)} + \underset{①}{2M \cdot \frac{\epsilon}{4M}} + \underset{③}{0}$$

$$= \epsilon .$$

Hence we have found step functions, $g \leqslant f \leqslant h$ on $[a,b]$ and $\int_a^b (h-g) < \epsilon$ (Ex _You do it?) so f is Riemann integrable over $[a,b]$. $\square$

The converse is in all the books (hand out?).