8: Sets of Measure Zero

8.1 Definition

A <u>null set</u> or a <u>set of measure zero</u> is quite simply a set E for which $\mu E = 0$.

8.2 Corollary

E is a null set if and only if $\forall \epsilon > 0$ there is a sequence of intervals (I_n) with $\sum_n \mu I_n < \epsilon$ and $E \subseteq \bigcup_n I_n$.

<u>Proof</u> $\emptyset \subseteq E$, $\emptyset$ is bounded and $\mathbb{R} \setminus \emptyset = \mathbb{R}$ is an outer set, so $\emptyset$ is an <u>inner</u> set.

Now $\bigcup_n I_n$ is an <u>outer</u> set and

$$\emptyset \subseteq E \subseteq \bigcup_n I_n \quad \text{with } \mu\left(\bigcup_n I_n \setminus \emptyset\right) < \epsilon.$$

By 3.5 E is integrable, and $\mu E = 0$, clearly.

$\square$

The following result shows that a set can be quite "densely distributed" and still be a null set.

8.3 Proposition

Let $E \subseteq \mathbb{R}$ be countable. Then E is a null set.

Proof

Let $E = \{x_1, x_2, \ldots\}$ let $I_n = (x_n - \epsilon/2^{n+1}, x_n + \epsilon/2^{n+1})$

then $\mu I_n = \epsilon/2^n$, $\bigcup_n I_n$ is of measure

less than $\sum_{n=1}^{\infty} \mu I_n = \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon$ and $E \subseteq \bigcup_n I_n$

The result follows from 8.2. □

8.4 Corollary

$\mathbb{Q}$ is a null set in $\mathbb{R}$!

Proof $\mathbb{Q}$ is countable .

Quite surprising !?

However a set may be uncountable and still a null set.

8.5 Example : The Cantor "Middle Third" Set

(Joke : That's exactly what it's not)

Consider $[0,1]$, let $I_1 = (1/3, 2/3)$,

$I_2 = (1/9, 2/9) \cup (7/9, 8/9)$, $I_3 = (1/27, 2/27) \cup (7/27, 8/27) \cup (13/27, 14/27)$

$\cup (25/27, 26/27)$,

and so on. This construction is best expressed as follows, I_1 is the "open middle third" of $[0,1]$, Now throw away . We are left with $[0, 1/3] \cup [2/3, 1]$.

Now throw away the open middle thirds of $[0, 1/3]$, $[2/3, 1]$ (i.e. I_2), they are $(1/9, 2/9)$ $(7/9, 8/9)$. This leaves

you with $[0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$, . . etc...

If you have done this n times then at the $(n+1)^{th}$ stage you throw away the open middle thirds of what was left at the n^{th} stage. The following should be clear after a moments thought; If what you throw away at the n^{th} stage is I_n then I_n is an open set (it is the union of finitely many (2^n) open intervals). Let $K = [0,1] \setminus \bigcup_n I_n$. Then as each I_n is open so is $\bigcup_n I_n$, hence K is closed, in $[0,1]$, and therefore bounded, i.e. it is compact. Now $[0,1] \setminus \bigcup_{n=1}^{\ell} I_n = K_\ell$ say, is a <u>basic</u> set and $K_\ell \supseteq K_{\ell+1}$, $K_\ell \downarrow K$ thus $\mu K = \lim_\ell \mu K_\ell$ <u>if</u> K is ~~meo~~ integrable

Since K_ℓ is a basic set, it is an <u>outer</u> set and

$$\mu K_\ell = \mu[0,1] - \mu(1/3, 2/3) - \mu(1/9, 2/9) - \mu(7/9, 8/9) - \dots$$
$$\dots - \mu\left(\frac{3^\ell - 1}{3^\ell}, 1\right)$$

$$= 1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27} - \dots - \frac{2^\ell}{3^\ell}$$

$$= 1 - \left(\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \dots + \frac{2^{\ell-1}}{3^\ell}\right)$$
$$= 1 - \frac{1}{3}\left(1 + 2/3 + 4/9 + \dots + 2^{\ell-1}/3^{\ell-1}\right)$$

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$\lim_{\ell} \mu K_\ell = 1 - \frac{1}{3}\left(\frac{1}{1-2/3}\right) = 0$ So by 8.2

$\mu K = 0$. Now we show K is uncountable

To see this one first needs to recall that every $x \in [0,1]$ has a ternary expansion (and how this is constructed (doi

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \qquad a_n \in \{0,1,2\} .$$

Now recall how we constructed K. we take $[0,1]$ anc throw away the open middle third $I_1 = (1/3, 2/3)$, if $y \in (1/3, 2/3)$ then its ternary expansion has $a_1 = 1$ and conversely.
So if $x \in [0,1] \setminus I_1$ then the ternary expansion of x does not have $a_1 = 1$, and conversely if the ternary expansio of x has $a_1 \neq 1$ then $x \in [0,1] \setminus I_1$. Removin the open middle thirds of $[0,1] \setminus I_1$ takes out just those x for which $a_2 = 1$ Removing the open middle thirds of $[0,1] \setminus (I_1 \cup I_2)$ will take out jus those x for which $a_3 = 1$, and so on. One can tart this up to be a rigorous mathematical induction which shows that

$$K = \{ x \in [0,1] : x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \text{ and } a_n \in \{0,2\} \} .$$

Now define $f : K \xrightarrow{\text{onto}} [0,1]$ by binary now

$$K \ni x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \longmapsto \sum_{n=1}^{\infty} b_n/2^n \quad \text{where } b_n = (1/2) a_n$$

One can check that this is a surjection, so K is uncountable. (Or just use a diagonal argument!

An interesting function

Following the construction of K, for any I_n let $\overline{I}_n$ denote its closure, so $\overline{I}_1 = [1/3, 2/3]$ and $\overline{I}_2 = [1/9\ 2/9] \cup [7/9\ 8/9]$ ie. the succesive <u>closed</u> middle thirds.

On $\overline{I}_1$ $\quad F(x) = 1/2$ $\qquad (1/2$

On $\overline{I}_2$ $\quad \underset{F(x)=1/4}{[1/9, 2/9]} \cup \underset{F(x)=3/4}{[7/9, 8/9]}$ $\qquad (1/4$

On $\overline{I}_3$ $\quad \underset{F(x)=1/8}{[1/27\ 2/27]} \cup \underset{F(x)=3/8}{[7/27\ 8/27]} \cup \underset{F(x)=5/8}{[19/27\ 20/27]} \cup \underset{F(x)=7/8}{[26/27\ 1]}$ $\qquad (1/8$

Take $\overline{I}_n$ it comprises 2^{n-1} closed intervals, define $F(x) = \frac{2k-1}{2^n}$ on the k^{th} (order by left end points as above).

Now any $x \in [0,1]$ has a ternary expansion $\sum_n \frac{a_n}{3^n} =$
Note that if $x_k = \sum_{n=1}^{k} a_n/3^n$ then $F(x_k)$ is defined by the induction above. Let $F(x) = \lim_{k \to \infty} F(x_k)$. One can check that F is <u>cts</u>, <u>increasing</u>, $F(0) = 0$, $F(1) = 1$,

and, (here the joke!), $F(x) = 0$ a.e. (well!, it's constant on each of the intervals you throw away from $[0,1]$ in the construction of K).