7: Measurable Sets and Measurable functions

We are already aware of some perfectly well behaved functio which fail to be integrable. For example $\chi_{\mathbb{R}}$ or x^2. Note with these two examples that if we "cut down" to a bounded interval, I, then we obtain integrable functions, i.e. $\chi_I \equiv \chi_{\mathbb{R}} \cdot \chi_I$ and $x^2 \chi_I(x)$ are in $\mathcal{L}$. It turns out that this observation has some real practical value and of considerable theoretical interest. Roughly speaking measurable functions are "well behaved" functions which fail to be integrabl because they are "too big near ∞" (warning; $1/x$ on $(\overset{1}{0},\infty)$ and zero elsewhere fails to be integrable but $1/x \to 0$ as $x \to \infty$)

7.1 Definitions

(i) For any $f : \mathbb{R} \to \mathbb{R}$ and $n \in \mathbb{N}$ define

$$f^{[n]}(x) = \begin{cases} f(x) & x \in [-n,n] \text{ and } |f(x)| \leq n \\ n & x \in [-n,n] \text{ and } f(x) > n \\ -n & x \in [-n,n] \text{ and } f(x) < -n \\ 0 & x \in \mathbb{R} \setminus [-n,n] \end{cases}$$

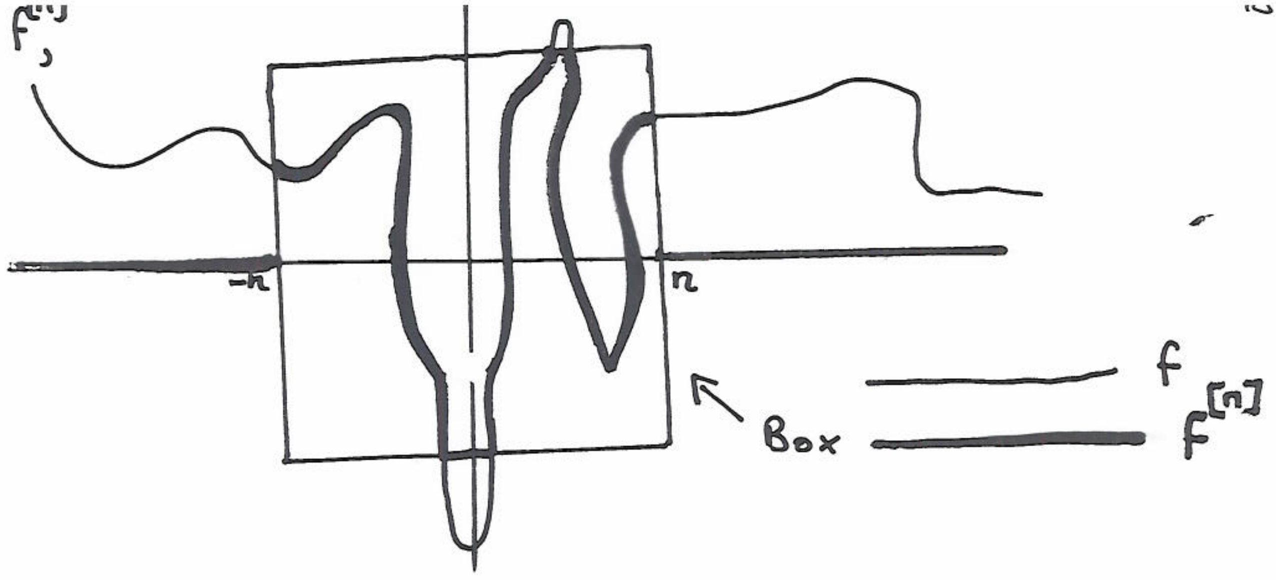
(ii) $f : \mathbb{R} \to \mathbb{R}$ is measurable $\iff \forall n \in \mathbb{N} \quad f^{[n]} \in \mathcal{L}$

(iii) A set $E \subseteq \mathbb{R}$ is measurable $\iff$ χ_E is a measurable function $\iff \forall n \quad [-n,n] \cap E$ is integrab

7.2 Remarks

(i) All that $f^{[n]}$ is, is f cut off in a box of sides n.

Picture of $f^{[n]}$,



(ii) $f^{[n]} = \left(f \wedge n\chi_{[-n,n]}\right) \vee \left(-n\chi_{[-n,n]}\right)$

7·3 Properties of Measurable Functions

Let f, g be measurable, $c \in \mathbb{R}$ then,

7·4 : Proposition

(i) $f + cg$ is measurable.

(ii) f^+, f^-, and $|f|$ are measurable.

(iii) $f \vee g$, $f \wedge g$ are measurable.

(iv) $f \cdot g$ (times) is measurable. (Warning $f \circ g$ may not be measurable)

(v) If $f \neq 0$ a.e. $1/f$ is measurable

(vi) If f_n is a sequence of measurable functions, $f_n \to f$ a.e. then f is measurable.

Proof A rather long exercise, don't do (iv), (v) yet.

7·5 Proposition

(i) Any continuous function is measurable

(ii) Any integrable function is measurable

Proof Exercises : DO THEM !

Here is a very useful result,

7·6 Theorem (Dominated Convergence)

If $(f_n) \subset \mathcal{L}$ and $f_n \to f$ a.e. and $\exists\, g \in \mathcal{L}^+$:

$\forall n \; |f_n| \leqslant g$ then $f \in \mathcal{L}$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f - f_n| \longrightarrow 0$$

and hence $\displaystyle \lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$.

Proof (HARD)

Let $k \in \mathbb{N}$ be fixed and $f_{kn} = \max\{f_k, f_{k+1}, \dots, f_n\}$ for $n \geqslant k$. Then $f_k \leqslant f_{kn} \leqslant f_{kn+1} \leqslant g$ by hypothesis.

Now $\int_{\mathbb{R}} f_{kn} \leqslant \int_{\mathbb{R}} g < \infty$ and $(f_{kn})_{n=k}^{\infty}$ is an increasing sequence in $\mathcal{L}$, so by Monotone convergence $h_k = \sup_{n \geqslant k} f_{kn} \in \mathcal{L}$

Now $h_{k+1} = \sup_{n \geqslant k+1} f_{k+1\,n}$, but $f_{k+1\,n} = \max\{f_{k+1}, \dots, f_n\}$ and so $f_{k+1\,n} \leqslant f_{kn}$ and hence $h_{k+1} \leqslant h_k$. Now by hypothesis $-g \leqslant f_k \leqslant h_k$ and so $\int_{\mathbb{R}} -g \leqslant \int_{\mathbb{R}} h_k$, so (h_k) is decreasing and $-\infty < \inf_k \int_{\mathbb{R}} h_k$. It follows from Monotone convergence that $h = \lim_k h_k \in \mathcal{L}$ and $\int_{\mathbb{R}} h = \inf_k \int_{\mathbb{R}} h_k$

What has this h to do with f ??.

Let x be fixed and $\epsilon > 0$. Disregarding a null set of x for which $f_n(x) \not\to f(x)$ we have, $\exists\, n(x,\epsilon) \in \mathbb{N} : n \geq n(x,\epsilon)$

$\Rightarrow \quad f(x) - \epsilon \leqslant f_n(x) \leqslant f(x) + \epsilon$. If $k \geqslant n(x,\epsilon)$ and $n \geqslant k$ then

$$f(x) - \epsilon \leqslant f_{kn}(x) \leqslant f(x) + \epsilon$$

and so $\quad f(x) - \epsilon \leqslant h_k(x) \leqslant f(x) + \epsilon$

thus $\quad f(x) - \epsilon \leqslant h(x) \leqslant f(x) + \epsilon$

Hence $\quad h(x) = f(x)$ a.e. ! . So $f \in \mathcal{L}$.

By arguments similar to that above, if we define $u_k = \inf_{n \geqslant k} f_n$ then $u_k \uparrow$ as $k \to \infty$ and $u = \lim_k u_k \in \mathcal{L}$, indeed $u = f$ a.e.

So we have,

$$u_k \leqslant f \leqslant h_k \qquad \text{a.e.} \qquad (\text{this is } f \text{ not } f_k)$$

$$u_k \leqslant f_n \leqslant h_k \qquad \forall n \geqslant k$$

So that $\quad 0 \leqslant |f - f_n| \leqslant h_k - u_k \quad$ a.e. for $n \geqslant k$

$$\left[\; h_k - u_k = (h_k - f) + (f - f_n) + (f_n - u_k) \geqslant f - f_n \quad \text{for } n \geqslant k \ (\text{a.e.}) \right.$$

$$= (h_k - f_n) + (f_n - f) + (f - u_k) \geqslant f_n - f \quad \text{for } n \geqslant k \ (\text{a.e.})$$

+ve

+ve

And as $h_k \downarrow$ and $u_k \uparrow$ to f a.e. we must have $\int_{\mathbb{R}} (h_k - u_k) \downarrow 0$ as $k \to \infty$. So $0 \leqslant \int_{\mathbb{R}} |f - f_n| \leqslant \int_{\mathbb{R}} (h_k - u_k)$

for $n \geq k$ gives the result. Finally

$$\left| \int_{\mathbb{R}} f - \int_{\mathbb{R}} f_n \right| \leq \int_{\mathbb{R}} |f - f_n| \longrightarrow 0 \text{ as } n \to \infty . \quad \square .$$

7.7 Lemma

Suppose f is integrable and $E \subseteq \mathbb{R}$ is integrable. Then $f\chi_E \in \mathcal{L}$.

Proof

Assume $f \geqslant 0$. Suppose that (f_n) is a sequence of integrable upper functions, $f_n \geqslant f$ and $\int_{\mathbb{R}} f_n < \int_{\mathbb{R}} f + 1/n$ then (Ex) we can construct a sequence of integrable upper functions (g_n) that decrease, $g_n \geqslant f$ and $\int_{\mathbb{R}} g_n < \int_{\mathbb{R}} f + 1/n$ by taking $g_1 = f_1$, $g_2 = f_1 \wedge f_2$, $\ldots$ $g_n = \bigwedge_{i=1}^{n} f_i$, $\ldots$. Since $h_n = g_n - f \in \mathcal{L}$, $h_n \downarrow 0$ and $h_n \geqslant 0$ then $\lim_n h_n \in \mathcal{L}$ and $\int_{\mathbb{R}} h = \lim_n \int_{\mathbb{R}} h_n = \lim_n \int_{\mathbb{R}} g_n - \int_{\mathbb{R}} f = 0$. So that $h = 0$ a.e. . Thus $g_n \downarrow f$ a.e. . Similarly if E_n are outer sets $E_n \downarrow$ and $\mu(E_n \setminus E) \longrightarrow 0$ then (by considering χ_{E_n} and χ_E and the argument above) $\chi_{E_n} \downarrow \chi_E$. Now by 4:18 (see remark following) $g_n \chi_{E_n}$ is an integrable upper function, $g_n \chi_{E_n} \downarrow f\chi_E$ and $\inf_n \int_{\mathbb{R}} g_n \chi_{E_n} > -\infty$. So by Monotone convergence $f\chi_E \in \mathcal{L}$.

Remark If f is not possitive then consider $f^{\pm}$, they are, $f^{\pm}\chi_E \in \mathcal{L}$ and $f\chi_E = f^{+}\chi_E - f^{-}\chi_E$. $\square$

8 Corollary (Of dominated convergence theorem)

Suppose f is a measurable function and $E \subseteq \mathbb{R}$ is integrable. If $f \restriction E$ is <u>bounded a.e.</u>, i.e. $\exists M > 0$: for a.e. $x \in E$ $|f(x)| \leq M$, then $f\chi_E \in \mathcal{L}$.

<u>Proof</u> Take $f \geq 0$ first

Since $f^{[n]} \uparrow f$ a.e. then $f^{[n]}\chi_E \uparrow f\chi_E$ a.e. <u>and</u> $f^{[n]} \in \mathcal{L}$ and E is integrable. So $f^{[n]}\chi_E \in \mathcal{L}$ by lemma 7.

Now $|f^{[n]}\chi_E| \leq \hat{M}$ where $\hat{M} = \begin{cases} 0 \text{ if } x \notin E \\ M \text{ if } x \in E \text{ and } f^{[n]}(x) \uparrow f \\ \infty \text{ if } x \in E \text{ and } f^{[n]}(x) \uparrow \infty \end{cases}$

Now $\hat{M} = M\chi_E$ a.e. ($\hat{M} = \infty$ on the set of points where $f^{[n]}\chi_E \not\to f\chi_E$) so $\hat{M} \in \mathcal{L}$. Now use 7.6 to conclude $f\chi_E \in \mathcal{L}$. $\square$

<u>Remark</u> If f is bounded a.e. on $\mathbb{R}$, i.e. $\exists M > 0$: $|f(x)| \leq M$ for a.e. $x \in \mathbb{R}$, <u>and</u> f is measurable then $f\chi_E$ is integrable for every integrable $E \subseteq \mathbb{R}$.

7.9 Definition

Let f be a measurable function and suppose that $E \subseteq \mathbb{R}$ is a measurable set <u>with</u> $f\chi_E \in \mathcal{L}$. We define the <u>integral of f over E</u> by

$$\int_E f = \int_{\mathbb{R}} f\chi_E .$$

We note that any integrable function is integrable over any integrable set, in fact

<u>Theorem 7:10</u> (This proof is "wrong" replace E_n with $E \cap E_n$)

Let $E \subseteq \mathbb{R}$ be measurable and $f \in \mathcal{L}$. Then $f\chi_E \in \mathcal{L}$

<u>Proof</u> Let $E_n = E \cap [-n, n]$, E_n is integrable and $f\chi_{E_n} \longrightarrow f\chi_E$ pointwise (a.e.). Now by 7.7 $f\chi_{E_n} \in \mathcal{L}$ and clearly $|f\chi_{E_n}| \leqslant |f\chi_E| \leqslant |f| \in \mathcal{L}$. So by 7.6 $f\chi_E \in \mathcal{L}$. $\square$.

<u>Remark</u> This proof shows that if h is measurable and $g \in \mathcal{L}$ and $|h| \leqslant g$ a.e. then $h \in \mathcal{L}$.

✳ 7:12 Theorem

Let $f : \mathbb{R} \to \mathbb{R}$. Then f is measurable if and only if $\forall c \in \mathbb{R}$, $\{f \geq c\}$ is measurable.

Proof

If f is measurable then (7.4) so are the functions

$$f_n(x) = n\big(f(x) \wedge c\big) - n\big(f(x) \wedge (c - 1/n)\big).$$

If $x \in \{x : f(x) \geq c\}$ then $f(x) \wedge c = c$ $\quad f(x) \wedge (c - 1/n) = c -$ so $f_n(x) = 1$. If $x \in \{x : f(x) < c\}$ then for large enough n, $c - 1/n \geq f(x)$ and so $f(x) \wedge (c - 1/n) = f(x)$ and $f_n(x) = 0$. All of which goes to show

$$f_n(x) \longrightarrow \chi_{\{f \geq c\}}(x) \quad \text{as } n \to \infty.$$

Again 7.4 (vi) shows $\chi_{\{f \geq c\}}$ is measurable.

If $\forall c \in \mathbb{R}$ $\chi_{\{f \geq c\}}$ (†) is measurable then $\{f > c\}$ is measurable too because

(†) why $\chi_{\{\ \}}$ measurab $\Rightarrow \{\ \}$ measurab

$$\{f > c\} = \bigcup_{n=1}^{\infty} \{f \geq c + 1/n\}$$

and a countable union of measurable sets is measurable.

Hence for $c \in \mathbb{R}$ and $\epsilon > 0$ $\{c \leq f < c+\epsilon\}$ is measurable because it is just $\{f \geq c\} \setminus \{f \geq c+\epsilon\}$ and the difference of measurable sets is measurable.

Now consider $f^{[n]}$. Define

$$g_k = \left(\sum_{\ell=0}^{(2n \cdot 2^k - 1)} \left(-n + \ell/2^k\right) \chi_{S_\ell} \right) + n\chi_{\{f \geq n\} \cap [-n,n]}$$

where $S_\ell = \{ -n + \ell/2^k \leq f < -n + (\ell+1)/2^k \} \cap [-n,n]$.

Now as $\{-n + \ell/2^k \leq f < -n + (\ell+1)/2^k\}$ is measurable then S_ℓ is <u>integrable</u>, as is $\{f \geq n\} \cap [-n,n]$, hence g_k is a linear combination of integrable functions, and is therefore integrable. Moreover $\forall k$ $|g_k| \leq n\chi_{[-n,n]}$ and $g_k \to f^{[n]}$ pointwise.

By dominated convergence $f^{[n]} \in \mathcal{L}$ and f is measurable.

$\square$