6 : The Normed Space L^1

We have already seen that any function equal a.e. to an element of $\mathcal{L}$ is indeed an element of $\mathcal{L}$ and the integrals coincide. Another, (slightly stronger?) remark we have made is that one can alter an integrable function on a set of measure zero without affecting its integral. So far as integration theory is concerned functions equal a.e. are the same! This observati[on] is given formal recognition as follows.

6·1 Definition (i) Define a relation on $\mathcal{L}$ by $f \sim g \Leftrightarrow f = g$ a[.e.]

Recall that this splits $\mathcal{L}$ up into disjoint equivalence classes. We will denote the class to which f belongs by $[f]$ for now. Later we will drop the box $[\cdot]$ and just think of functions equal a.e. as being identified so that we think of the equiva-lence classes as functions.

(ii) For $f, g \in \mathcal{L}$ define $[f] + [g] = [f+g]$

$f \in \mathcal{L}, k \in \mathbb{R}$ define $k[f] = [kf]$

(iii) For $f \in \mathcal{L}$, $\|[f]\|_1 = \int_{\mathbb{R}} |f|$

Hand out on Banach Spaces?

6·2 Proposition

Let $L^1 = \{[f] : f \in \mathcal{L}\}$ with the operations given by 6·1(ii) and the metric given by 6·1(iii) i.e. $d([f],[g]) = \|[f]-[g]\|_1$. Then L^1 is a complete normed vector space.

Proof You can check the following details;

(i) The operations of addition and scalar multiplication are well defined.

(ii) $[f] \longmapsto \|[f]\|_1$ is a norm. (See hand out).

Try to stop here, give exercises, & resume a few days later.

We show that L^1 is complete. To simplify the notation we shall think of the equivalence classes in L^1 just in terms of a representative of the class. So we will <u>write</u> $\|f - g\|_1$ but we <u>mean</u> $\|[f]-[g]\|_1$, in fact no confusion should occour because the proof to follow works just as well for functions as it does for classes, it just needs slight changes of language at a few points.

<u>Step 1</u>

We have $(f_n) \subset L^1$ and $\forall \epsilon > 0 \ \exists N_\epsilon \in \mathbb{N} : m,n \geq N_\epsilon$

$$\Rightarrow \|f_n - f_m\|_1 < \epsilon.$$

Let k_1 = least k : $\|f_m - f_n\|_1 < 1/2^1$ for $m,n \geq k$

k_2 = least k, $k > k_1$: $\|f_m - f_n\|_1 < 1/2^2$ for $m,n \geq k$

$\vdots$

k_p = least k, $k > k_{p-1}$: $\|f_m - f_n\|_1 < 1/2^p$ for $m,n \geq k$

$\vdots$

Consider the subsequence (f_{k_n}). Then $\|f_{k_{n+1}} - f_{k_n}\|_1 < 1/2^n$

and $\sum_{n=1}^{\infty} \|f_{k_{n+1}} - f_{k_n}\|_1 < 1 < \infty$.

NB. This is saying that $\sum_{n=1}^{\infty} \int_{\mathbb{R}} |f_{k_{n+1}} - f_{k_n}| < \infty$ thinking in terms of functions, so we can use 5·9.

By 5·9, (thinking of representatives) $\sum_n (f_{k_{n+1}} - f_{k_n})(x)$ converges a.e. to an integrable function. Now the partial sums

$$\sum_{n=1}^{m} (f_{k_{n+1}} - f_{k_n})(x) = f_{k_{m+1}}(x) - f_{k_1}(x).$$

Hence $\lim_m \left(f_{k_{m+1}}(x) - f_{k_1}(x) \right) = \lim_m \left(f_{k_{m+1}}(x) \right) - f_{k_1}(x)$ defines an integrable function. But $f_{k_1} \in \mathcal{L}$ so this shows that $\lim_m f_{k_{m+1}}(x)$ defines an integrable function. Let $g = \lim_m f_{k_{m+1}}$, for those x for which the limit exists.

<u>Step 2</u>

$$\| g - f_n \|_1 \leq \| g - f_{k_{m+1}} \|_1 + \| f_{k_{m+1}} - f_n \|_1$$

$$< \| g - f_{k_{m+1}} \|_1 + 1/2^{m+1} \quad \text{if } n \geq k_{m+1}$$

moduli

Now

$$|g - f_{k_{m+1}}| = \lim_{r \to \infty} | f_{k_r} - f_{k_{m+1}} |$$

$$= \lim_{r \to \infty} \left| \sum_{s=m+1}^{r} (f_{k_{s+1}} - f_{k_s}) \right|$$

$$\leq \lim_{r \to \infty} \sum_{s=m+1}^{r} | f_{k_{s+1}} - f_{k_s} |$$

and this last term is known to converge a.e., so by Monotone Convergen[ce]

$$\int_{\mathbb{R}} |g - f_{k_{m+1}}| \leq \int_{\mathbb{R}} \lim_{r \to \infty} \sum_{s=m+1}^{r} |f_{k_{s+1}} - f_{k_s}|$$

$$= \lim_{r \to \infty} \int_{\mathbb{R}} \sum_{s=m+1}^{r} |f_{k_{s+1}} - f_{k_s}|$$

back to equiclasses

$$= \sum_{s=m+1}^{\infty} \int_{\mathbb{R}} |f_{k_{s+1}} - f_{k_s}| = \sum_{s=m+1}^{\infty} \| f_{k_{s+1}} - f_{k_s} \|_1$$

But $\sum_{s=m+1}^{\infty} \| f_{k_{s+1}} - f_{k_s} \|_1 < \sum_{s=m+1}^{\infty} 1/2^s < 1/2^m$.

So $\| g - f_{k_{m+1}} \|_1 < 3/2^{m+1}$. So to show $f_n \to g$ in $\|\cdot\|_1$ <u>First</u> make $\| g - f_{k_{m+1}} \|_1$ <u>small</u> <u>then</u> adjust n so that $\| f_{k_{m+1}} - f_n \|_1$ is small. I leave you to do the rigorous "ϵ argument".

$\square$