5 Properties of Integrable Functions

5·1 Theorem

If $f \in \mathcal{L}$ and $f = g$ a.e. then $g \in \mathcal{L}$ and $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$

Proof

Let $N = \{x : f(x) \neq g(x)\}$. Let $\epsilon > 0$, by 4:17 there is an upper function h : $h \restriction N = \infty$, $h \geq 0$, $\int_{\mathbb{R}} h < \epsilon/2$. Let k be upper, $f \leq k$, $\int_{\mathbb{R}} k < \int_{\mathbb{R}} f + \epsilon/2$. Then $h+k$ is upper

$h+k \geqslant g$ and $\int_{\mathbb{R}}(h+k) = \int_{\mathbb{R}} h + \int_{\mathbb{R}} k < \frac{\epsilon}{2} + \int_{\mathbb{R}} f + \epsilon/2$.

Hence $\int_{\mathbb{R}}^{*} g < \int_{\mathbb{R}} f + \epsilon$. Now choose a lower function ℓ such that $\ell \leqslant f$ and $\int_{\mathbb{R}} \ell > \int_{\mathbb{R}} f - \epsilon/2$, by 4:15 we have

$\int_{\mathbb{R}} \ell \leqslant \int_{\mathbb{R}} h + k$. So observe that $-h$ is a lower function, $\ell - h$ is a lower function, $\ell - h \leqslant g$ so $\int_{\mathbb{R}}(\ell - h) \leqslant \int_{*\mathbb{R}} g$

But $\int_{\mathbb{R}}(\ell-h) = \int_{\mathbb{R}} \ell - \int_{\mathbb{R}} h > \int_{\mathbb{R}} f - \epsilon/2 - \frac{\epsilon}{2}$. So

$$\int_{*\mathbb{R}} g > \int_{\mathbb{R}} f - \epsilon. \text{ Thus } \int_{*\mathbb{R}} g = \int_{\mathbb{R}}^{*} g = \int_{\mathbb{R}} f. \quad \square$$

5·2 Corollary

If $f \in \mathcal{L}$ and $f = 0$ a.e. then $\int_{\mathbb{R}} f = 0$.

Proof

Take $g(x) = 0 \ \forall x$ in 5·1, (g is clearly in $\mathcal{L}$ & $\int_{\mathbb{R}} g = 0$). $\square$

5·3 Theorem

Let $f \in \mathcal{L}$. If $f \geqslant 0$ a.e. and $\int_{\mathbb{R}} f = 0$ then $f = 0$ a.e. .

Proof

Let $\epsilon > 0$. Let $n \in \mathbb{N}$, choose an upper function $g_n \geqslant f$ with $\int_{\mathbb{R}} g_n < \frac{1}{2^n}$

Let $g = \lim_m \sum_{n=1}^{m} g_n$. Since we only know $f \geqslant 0$ a.e. we

cannot assume $\left(\sum_{n=1}^{m} g_n\right)$ is increasing, to rectify this we use 4·17.

Let $N = \{x : f(x) \ngeq 0\}$ and let h_n be an upper function :

$h_n \uparrow H = \infty$, $h_n \geqslant 0$, $\int_{\mathbb{R}} h_n < \frac{\epsilon}{2^n}$. Consider $g_n + h_n$, then $g_n + h_n$ is an

upper function + $\int_{\mathbb{R}} (g_n + h_n) < \frac{1}{2^n} + \frac{\epsilon}{2^n}$. Let $G = \lim_m \sum_{n=1}^{m} (g_n + h_n)$.

Then G is an upper function and $\int_{\mathbb{R}} G = \lim_m \int_{\mathbb{R}} (g_n + h_n)$ by 4.16

So $\int_{\mathbb{R}} G = 1 + \epsilon$. Since G is an integrable upper function

it is finite almost everywhere, that is, $G(x) < \infty$ for almost

all x . Now $G(x) = \sum_{n=1}^{\infty} g_n(x) + h_n(x) \geqslant \sum_{n=1}^{\infty} f(x)$ (ie. $f(x) + f(x) + \ldots$

So $f(x) = 0$ for almost all x . $\square$.

5·4 Corollary

If $f \in \mathcal{L}$ then $f = 0$ a.e. $\Longleftrightarrow \int_{\mathbb{R}} |f| = 0$.

Proof By 5·2 and 5·3. $\square$.

Remark.

Before we proceed to 5·5 we need to introduce a technical device made necessary by our allowing functions to take non finite values. As you know in $\mathbb{R}^*$ you cannot form $\infty - \infty$, this is an "illegal" move. But if f and g are integrable functions it is perfectly reasonable that there may exist $x \in \mathbb{R}$ for which $f(x) = \infty$ and $g(x) = -\infty$. This poses a problem; What is $f(x) + g(x)$?, i.e., how can we add the functions f and g together to form another function defined on <u>all</u> of $\mathbb{R}$?. The answer to this problem is to define $f(x) + g(x) = \infty$ at all points x where f and g take differing but infinite values. At all other points you just use the usual operations in $\mathbb{R}^*$. With this definition you can manipulate sums of functions with impunity.

You might argue that it would be better to define $f(x) + g(x) = 0$. But look, if h also takes an infinite value at x,

$$(f(x) + g(x)) + h(x) = (\infty - \infty) + \infty = 0 + \infty = \infty.$$

but then

$$h(x) + (f(x) + g(x)) \stackrel{?}{=} (h(x) + f(x)) + g(x)$$

$$= (\infty + \infty) - \infty$$

$$= \infty - \infty$$

$$= 0$$

But surely

$$\big(f(x) + g(x)\big) + h(x) = h(x) + \big(f(x) + g(x)\big)$$

$$= \big(h(x) + f(x)\big) + g(x) \quad ?$$

Well obviously <u>not</u> if $\infty - \infty = 0$!. So you bugger up the algebraic operations with the definiti[on] $\infty - \infty = 0$. Note however that, with our stated definition

Space out some

$$\big(f(x) + g(x)\big) + h(x) = (\infty - \infty) + \infty = \infty + \infty = \infty$$

$$f(x) + \big(g(x) + h(x)\big) = \infty + (-\infty + \infty) = \infty + \infty = \infty$$

So sums associate, and,

$$f(x) + g(x) = \infty + (-\infty) = \infty$$

$$g(x) + f(x) = (-\infty) + \infty = \infty$$

So sums commute.

5.5 Theorem

Let $f, g, \ldots, h \in \mathcal{L}$ and $\alpha, \beta \ldots \in \mathbb{R}$. Then

(i) $\alpha f + \beta g \in \mathcal{L}$, and $\int_{\mathbb{R}} (\alpha f + \beta g) = \alpha \int_{\mathbb{R}} f + \beta \int_{\mathbb{R}} g$.

(ii) $f \leq g \Rightarrow \int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$.

(iii) $f \in \mathcal{L} \Rightarrow |f|, f^+, f^- \in \mathcal{L}$.

(iv) $f, g \in \mathcal{L} \Rightarrow \mathcal{L} \ni f \vee g \equiv x \mapsto \max\{f(x), g(x)\}$ (picture please)

$\mathcal{L} \ni f \wedge g \equiv x \longmapsto \min\{f(x), g(x)\}$

Proof (i) Exercise

(ii) Exercise

(iii) Exercise

(iv) From (iii)&(i) $f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$

$f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$

The following result ties together our treatment of the measure of sets and the integrals of functions

5.6 (Big) Theorem

Let $E \subseteq \mathbb{R}$ then E is integrable $\Leftrightarrow \chi_E \in \mathcal{L}$.

Moreover $\mu E = \int_{\mathbb{R}} \chi_E$ whenever E is integrable.

Proof {Let $\epsilon > 0$,
{If E is integrable $\exists$ G outer and H inner with $G \supseteq E \supseteq H$ and $\mu(G \setminus H) < \epsilon$. Let $g = \chi_G$ and $h = \chi_H$ then (Ex g is upper and h is lower and $\int_{\mathbb{R}} g = \mu G$, $\int_{\mathbb{R}}^{*} h = \mu H$ and $\int_{\mathbb{R}} g - \int_{\mathbb{R}} h = \mu(G \setminus H) < \epsilon$, thus $\int_{*\mathbb{R}} \chi_E = \int_{\mathbb{R}}^{*} \chi_E = \mu E$.

Conversely if χ_E is integrable $\exists$ g upper and k lower, $k \leq \chi_E \leq g$ and $\int_{\mathbb{R}}(g-k) < \epsilon/2$. Let g be the limit of the increasing sequence of step functions (u_n). If $G_n = \{x : u_n(x) \geq 1\}$ then G_n is a basic set ( ) and hence $G = \bigcup_n G_n$ is an outer set. Clearly $E \subseteq G$. Let $K = \{x : k(x) \geq \frac{1}{2}\}$ then $\mathbb{R}\setminus K = \{x : k(x) < \frac{1}{2}\}$ $= \{x : -k(x) > -\frac{1}{2}\}$ which is an outer set (Ex). Since K is bounded (Ex) it follows that K is an inner set. Clearly $K \subseteq E$. Now $G\setminus K$ is an outer set and $g-k$ is an outer function both are 'integrable' so $(g-k)\chi_{G\setminus K}$ is an integrable outer function. Now on $G\setminus K$ $g-k \geq \frac{1}{2}$ so

$$(g-k)\chi_{G\setminus K} \geq \tfrac{1}{2}\chi_{G\setminus K}$$

(nb. integrable upper fns are integrable in the sense of § 4·13 (iv)

hence $$\frac{\epsilon}{2} > \int_{\mathbb{R}} g-k \overset{4{\cdot}18}{\geq} \int_{\mathbb{R}}(g-k)\chi_{G\setminus K} \underset{(ii)}{\overset{5{\cdot}5}{\geq}} \int_{\mathbb{R}} \tfrac{1}{2}\chi_{G\setminus K} \underset{(i)}{\overset{5{\cdot}5}{=}} \tfrac{1}{2}\int_{\mathbb{R}}\chi_{G\setminus K}$$

$$\overset{4{\cdot}18}{=} \tfrac{1}{2}\mu G\setminus K$$

So $\mu G\setminus K = \mu G - \mu K < \epsilon$. This shows E is integrable. $\square$

This proof is wrong

Take $G_n = (u_n \geq 3/4)$ $K = (k \geq 1/4)$ etc..

5.5 Theorem

Let $f, g \dots, h \in \mathcal{L}$ and $\alpha, \beta, \dots \in \mathbb{R}$. Then

(i) $\alpha f + \beta g \in \mathcal{L}$ and $\int_{\mathbb{R}} (\alpha f + \beta g) = \alpha \int_{\mathbb{R}} f + \beta \int_{\mathbb{R}} g$

(ii) $f \leqslant g \Rightarrow \int_{\mathbb{R}} f \leqslant \int_{\mathbb{R}} g$

Proof

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Proof

(i) Let $f \in \mathcal{L}$ and $\alpha \in \mathbb{R}$. If $\alpha > 0$ and u is upper, ℓ lower and, $\ell \leqslant f \leqslant u$ and $\epsilon > 0$ we can choose u, ℓ so that

$$\int_{\mathbb{R}} u < \int_{\mathbb{R}} f + \frac{\epsilon}{\alpha} \quad , \quad \int_{\mathbb{R}} f < \int_{\mathbb{R}} \ell + \epsilon/\alpha .$$

But then $\alpha\ell \leqslant \alpha f \leqslant \alpha u$ and $\alpha\ell$ is lower αu is upper. ~~Now if $\epsilon > 0$ we can choose u, ℓ so that~~

$$\int_{\mathbb{R}} \alpha u = \alpha \int_{\mathbb{R}} u < \alpha \int_{\mathbb{R}} f + \frac{\alpha \epsilon}{\alpha} = \alpha \int_{\mathbb{R}} f + \epsilon$$

$$\alpha \int_{\mathbb{R}} f < \alpha \int_{\mathbb{R}} \ell + \frac{\alpha \epsilon}{\alpha} = \int_{\mathbb{R}} \alpha \ell + \epsilon$$

So that

$$\int_{\mathbb{R}}^{*} \alpha f < \alpha \int_{\mathbb{R}} f + \epsilon < \int_{*\mathbb{R}} \alpha f + 2\epsilon$$

or

$$\int_{\mathbb{R}}^{*} \alpha f < \alpha \int_{\mathbb{R}} f < \int_{*\mathbb{R}} \alpha f + \epsilon \leqslant \int_{\mathbb{R}}^{*} \alpha f + \epsilon .$$

Since $\epsilon > 0$ is arbitrary

$$\int_{\mathbb{R}}^{*} \alpha f = \int_{*\mathbb{R}} \alpha f = \alpha \int_{\mathbb{R}} f$$

and $\alpha f \in \mathcal{L}$.

If $\alpha = 0$ the result is obvious.

If $\alpha < 0$ and u, l, ϵ are as above then

$$\alpha u \geqslant \alpha f \leqslant \alpha l$$

but αu is now <u>lower</u>, αl now <u>upper</u> and

$$\alpha \int_{\mathbb{R}} u > \alpha \int_{\mathbb{R}} f + \epsilon \quad , \quad \alpha \int_{\mathbb{R}} f > \alpha \int_{\mathbb{R}} l + \epsilon .$$

Once again, $\alpha \int_{\mathbb{R}} u = \int_{\mathbb{R}} \alpha u$ and $\alpha \int_{\mathbb{R}} l = \int_{\mathbb{R}} \alpha l$

so that

$$\begin{aligned} \int_{*\mathbb{R}} \alpha f \geqslant \int_{\mathbb{R}} \alpha u = \alpha \int_{\mathbb{R}} u &> \alpha \int_{\mathbb{R}} f + \epsilon \\ &> \alpha \int_{\mathbb{R}} l + 2\epsilon \\ &= \int_{\mathbb{R}} \alpha l + 2\epsilon \\ &\geqslant \int_{\mathbb{R}}^{*} \alpha f + 2\epsilon , \end{aligned}$$

Argue as above to get,

$$\int_{*\mathbb{R}} \alpha f = \int_{\mathbb{R}}^{*} \alpha f = \alpha \int_{\mathbb{R}} f$$

and $\alpha f \in \mathcal{L}$.

If $f, g \in \mathcal{L}$ then for $\epsilon > 0$ there are upper functions u_f, u_g, lower functions ℓ_f, ℓ_g with,

$$\ell_f \leq f \leq u_f$$

$$\ell_g \leq g \leq u_g$$

and

$$\int_{\mathbb{R}} \ell_f > \int_{\mathbb{R}} f - \frac{\epsilon}{2}, \quad \int_{\mathbb{R}} u_f < \int_{\mathbb{R}} f + \epsilon/2$$

$$\int_{\mathbb{R}} \ell_g > \int_{\mathbb{R}} g - \frac{\epsilon}{2}, \quad \int_{\mathbb{R}} u_g < \int_{\mathbb{R}} g + \epsilon/2$$

Now, bearing in mind our definition of the sum of possibly infinite functions,

$$\ell_f + \ell_g \leq f + g \leq u_f + u_g$$

$\ell_f + \ell_g$ is lower, $u_f + u_g$ is upper

and

$$\int_{\mathbb{R}} (\ell_f + \ell_g) = \int_{\mathbb{R}} \ell_f + \int_{\mathbb{R}} \ell_g > \int_{\mathbb{R}} f - \frac{\epsilon}{2} + \int_{\mathbb{R}} g - \frac{\epsilon}{2}$$

$$= \int_{\mathbb{R}} f + \int_{\mathbb{R}} g - \epsilon$$

$$\int_{\mathbb{R}} (u_f + u_g) = \int_{\mathbb{R}} u_f + \int_{\mathbb{R}} u_g < \int_{\mathbb{R}} f + \int_{\mathbb{R}} g + \epsilon$$

So $\displaystyle\int_{*\mathbb{R}} (f+g) = \int_{\mathbb{R}}^{*} (f+g) = \int_{\mathbb{R}} f + \int_{\mathbb{R}} g$. Thus $f + g \in \mathcal{L}$

(ii)

If $f \leqslant g$ and ℓ is lower $\ell \leqslant f$ and u is upper $g \leqslant u$ then by 4.15 we know $\int_{\mathbb{R}} \ell \leqslant \int_{\mathbb{R}} u$ because $u \geqslant \ell$. So taking the supremum of the left side of

$$\int_{\mathbb{R}} \ell \leqslant \int_{\mathbb{R}} u$$

over all lower $\ell \leqslant f$ gives

$$\int_{*\mathbb{R}} f \leqslant \int_{\mathbb{R}} u .$$

Taking the infimum of the right side of the last inequality over all upper $u \geqslant g$ gives

$$\int_{*\mathbb{R}} f \leqslant \int_{\mathbb{R}}^{*} g$$

But $f, g \in \mathcal{L}$ so $\int_{\mathbb{R}} f = \int_{*\mathbb{R}} f \leqslant \int_{\mathbb{R}}^{*} g = \int_{\mathbb{R}} g$.

5.6 (BIG) Theorem

Let $E \subseteq \mathbb{R}$. E is integrable $\iff$ $\chi_E \in \mathcal{L}$, ~~and~~ Moreover

$\mu E = \int_{\mathbb{R}} \chi_E$ whenever E is integrable.

Proof

Let $\epsilon > 0$. If E is an integrable set $\exists$ G outer and H inner with $G \supseteq E \supseteq H$, and $\mu(G \setminus H) < \epsilon$. Let $g = \chi_G$, $h = \chi_H$. Then g is upper and h lower because G outer means there are basic sets $G_n \uparrow G$ now χ_{G_n} is a step function and $\chi_{G_n}(x) \uparrow \chi_G(x)$ for $x \in \mathbb{R}$ making g an upper function. A similar argument works for h using the fact that H is the intersection of a decreasing sequence of basic sets. It is clear then that $\int_{\mathbb{R}} g = \mu G$ and $\int_{\mathbb{R}} h = \mu H$ while $\int_{\mathbb{R}} g - \int_{\mathbb{R}} h = \mu G - \mu H = \mu(G \setminus H) < \epsilon$.

Now $h \leq \chi_E \leq g$ and so

$$\int_{\mathbb{R}} h \leq \int_{*\mathbb{R}} \chi_E \leq \int_{\mathbb{R}}^{*} \chi_E \leq \int_{\mathbb{R}} g$$

Thus $\int_{\mathbb{R}}^{*} \chi_E - \int_{*\mathbb{R}} \chi_E \leq \int_{\mathbb{R}} g - \int_{\mathbb{R}} h < \epsilon$. Since $\epsilon > 0$ was arbitrary this shows χ_E is integrable. But what is $\int_{\mathbb{R}} \chi_E$? Well

$$\int_{\mathbb{R}} h = \mu H \leq \mu E \leq \mu G = \int_{\mathbb{R}} g$$

and $\int_{\mathbb{R}} h \leq \int_{\mathbb{R}} \chi_E \leq \int_{\mathbb{R}} g$

So that $\int_{\mathbb{R}} \chi_E - \mu E \leqslant \int_{\mathbb{R}} g - \int_{\mathbb{R}} h = \mu(G \setminus H) < \epsilon$

and we have $\int_{\mathbb{R}} \chi_E = \mu E$. Conversely let us suppose $\chi_E \in \mathcal{L}$. There are functions g (upper), k (lower),

$k \leqslant \chi_E \leqslant g$ and $\int_{\mathbb{R}} (g-k) < \epsilon/2$. If g is the pointwise limit of the increasing sequence of step funct
(u_n) and $G_n = \{x : u_n(x) \geqslant 3/4\}$ then G_n is a basic set, $G_n \subseteq G_{n+1}$, and so $G = \bigcup_n G_n$ is an outer set.
If $x \in E$ then $\chi_E(x) = 1 \leqslant g(x) = \lim_n u_n(x)$ and so $x \in \bigcup_n G_n$, i.e., $E \subseteq G$. Now let K

be $\{x : k(x) \geqslant 1/4\}$. Consider $\mathbb{R} \setminus K = \{x : k(x) < 1/4\}$

$= \{x : -k(x) > -\frac{1}{4}\}$ and as $-k$ is upper this set is outer (Exercise: If h is upper and $\alpha \in \mathbb{R}$, $\{x : h(x) > \alpha\}$ is outer
Now K is bounded (why?) and $\mathbb{R} \setminus K$ is outer, so K is inn

By 4. $G \setminus K$ is an outer set and as $g-k$ is integrable $(g-k)\chi_{G \setminus K}$ is integrable (by 4.18). So

$$\frac{\epsilon}{2} > \int_{\mathbb{R}} (g-k) \overset{4.18}{\geqslant} \int_{\mathbb{R}} \chi_{G \setminus K}(g-k) \overset{5.5(ii)}{\geqslant} \int_{\mathbb{R}} \tfrac{1}{2} \chi_{G \setminus K}$$

(because $g-k \geq 1/2$ on $G \setminus K$)

$$\overset{5.5(i)}{=} \tfrac{1}{2} \int_{\mathbb{R}} \chi_{G \setminus K} = \tfrac{1}{2} \mu(G \setminus K).$$

This shows E is an integrable set, for $G \supseteq E \supseteq K$, G is outer, K inner, and $\mu G - \mu K < \epsilon$

5·1 Theorem Lebesgues Monotone Convergence Theorem

Let (f_n) be a sequence of integrable functions with $f_n \leqslant f_{n+1}$, $n = 1, 2, \dots$ and $f = \lim_n f_n$ and $\sup_n \int_{\mathbb{R}} f_n < \infty$

Then $f \in \mathcal{L}$ and $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$.

Proof

For each n $\quad f_n \leqslant f$ and there is a lower function $\ell_n \leqslant f_n$ with $\int_{\mathbb{R}} f_n \geqslant \int_{\mathbb{R}} \ell_n > \int_{\mathbb{R}} f_n - \frac{1}{n}$, so that we must have

$$\lim_n \int_{\mathbb{R}} f_n \leqslant \int_{*\mathbb{R}} f \, .$$

Define $g_n = f_{n+1} - f_n \geqslant 0$, so that $\sum_{k=1}^{n} g_k = f_{n+1} - f_1$ and $\sum_{k=1}^{\infty} g_k = f - f_1$. By 5·5 (i) $g_n \in \mathcal{L}$ and

~~Now~~ $$\int_{\mathbb{R}} \left(\sum_{k=1}^{n} g_k \right) = \sum_{k=1}^{n} \int_{\mathbb{R}} g_k = \int_{\mathbb{R}} f_{n+1} - \int_{\mathbb{R}} f_1$$

hence $$\lim_n \int_{\mathbb{R}} \left(\sum_{k=1}^{n} g_k \right) = \lim_n \sum_{k=1}^{n} \int_{\mathbb{R}} g_k = \lim_n \left(\int_{\mathbb{R}} f_{n+1} \right) - \int_{\mathbb{R}} f_1 \, .$$

Let $\epsilon > 0$ and for each n let h_n be an integrable upper function satisfying

$$h_n \geqslant g_n \qquad \int_{\mathbb{R}} h_n < \int_{\mathbb{R}} g_n + \epsilon/2^n \, .$$

By 4·16 $\sum_1^{\infty} h_n$ is an upper function and

$$\int_{\mathbb{R}} \left(\sum_1^{\infty} h_n \right) = \sum_1^{\infty} \int_{\mathbb{R}} h_n \leqslant \sum_1^{\infty} \left(\int_{\mathbb{R}} g_n + \epsilon/2^n \right) = \lim_n \int_{\mathbb{R}} f_n - \int_{\mathbb{R}} f_1 + \epsilon \, .$$

So $\sum_1^\infty h_n \geq \sum_1^\infty g_n = f - f_1$ and $\forall \epsilon > 0, \int_{\mathbb{R}} \left(\sum_1^\infty h_n\right) < \lim_n \int_{\mathbb{R}} f_n - \int_{\mathbb{R}} f_1 + \epsilon$

this says that $\int^*_{\mathbb{R}} (f - f_1) \leq \int_{\mathbb{R}} \left(\sum_1^\infty h_n\right) \leq \lim_n \int_{\mathbb{R}} f_n - \int_{\mathbb{R}} f_1$.

Now
$$\int^*_{\mathbb{R}} f = \int^*_{\mathbb{R}} (f - f_1) + f_1 = \int^*_{\mathbb{R}} (f - f_1) + \int_{\mathbb{R}} f_1$$
$$\leq \lim_n \int_{\mathbb{R}} f_n$$
$$\leq \int_{*\,\mathbb{R}} f$$

And so $\int^*_{\mathbb{R}} f = \int_{*\,\mathbb{R}} f$ & $f \in \mathcal{L}$. $\square$

5.8 <u>Corollarys</u>

(i) Let (f_n) be a <u>decreasing</u> sequence of integrable functions with $\inf_n \int_{\mathbb{R}} f_n > -\infty$. Then $\lim_n f_n = f \in \mathcal{L}$ and $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$.

(ii) Let (g_n) be a sequence of integrable functions with $g_n \geq 0$ a.e. and $\sum_n \int_{\mathbb{R}} g_n < \infty$. Then $g = \sum_n g_n \in \mathcal{L}$ and $\int_{\mathbb{R}} g = \sum_n \int_{\mathbb{R}} g_n$.

<u>Proof</u> Exercise (application of 5.7), you will need 5.1 for (ii) perhaps

5.9 Theorem

Let $f \in \mathcal{L}$. Then $|f| \in \mathcal{L}$ as is f^+ and f^-.

Proof

Let u, v be upper functions, $u_n \in S$, $v_n \in S$ with $u_n \uparrow u$ and $v_n \uparrow v$. Then $u_n \wedge v_n \in S$ and $u_n \wedge v_n$ increases to $u \wedge v$. So the minimum of two upper functions is an upper function.

Let $f \in \mathcal{L}$ and $\forall n$ u_n be an upper function with $u_n \geq f$ and $\int_{\mathbb{R}} u_n < \int_{\mathbb{R}} f + 1/n$. Now define

$$h_1 = u_1, \quad h_2 = u_1 \wedge u_2, \quad h_3 = u_1 \wedge u_2 \wedge u_3, \ldots, \quad h_n = \bigwedge_{L=1}^{n} u_n, \ldots$$

then (h_n) is a decreasing sequence of upper functions, and

$$\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} h_n \leq \int_{\mathbb{R}} u_n \leq \int_{\mathbb{R}} f + 1/n \Big) \xrightarrow[\text{please jane}]{\text{Centre}}$$

So $\lim_n \int_{\mathbb{R}} h_n = \int_{\mathbb{R}} f$. But by the monotone convergence theorem $\lim_n h_n = h \in \mathcal{L}$ and clearly $h \geq f$, because $h_n \geq f$, and also $\int_{\mathbb{R}} (h - f) = 0$. So by 5. $h = f$ a.e..

Now $h_n \downarrow h$ pointwise so $h_n^+ \downarrow h^+$ pointwise.

Now h_n^+ is an upper function and $h_n^+ \downarrow$ and since $h_n^+ \geq 0$ $\forall n$ we must have $\inf \int_{\mathbb{R}} h_n^+ > -\infty$. Using monotone convergence we deduce that $h^+ = \lim_n h_n^+ \in \mathcal{L}$. Now $h(x) = f(x)$ for μ almost every $x \in \mathbb{R}$, by discarding a null set N we have $h(x) = f(x)$ for $x \in \mathbb{R} \setminus N$. So $h^+(x) = f^+(x)$ for $x \in \mathbb{R} \setminus N$ or $f^+ = h^+$ for μ almost every $x \in \mathbb{R}$.

But 5.1 now tells us that in this case $f^+ \in \mathcal{L}$. So $f \in \mathcal{L} \Rightarrow f^+ \in \mathcal{L}$. If $f \in \mathcal{L}$ and $f^+ \in \mathcal{L}$ then obviously $f^- \in \mathcal{L}$ and so $|f| = f^+ + f^- \in \mathcal{L}$ (by 5.5 (i) in each case).

5.~~9~~10 Theorem (VIT)

Suppose $(g_n) \subset \mathcal{L}$ and $\sum_n \int |g_n| < \infty$ then

(i) $\sum_n g_n(x)$ converges μ - a.e. and $g(x) = \begin{cases} \sum_n g_n(x) & \text{if C} \\ 0 & \text{if not C} \end{cases}$

is an integrable function.

(ii) $$\int_{\mathbb{R}} g = \int_{\mathbb{R}} \sum_n g_n = \sum_n \int_{\mathbb{R}} g_n .$$

Proof $(|g_n|)$ is a sequence of non negative integrable functions ($\overset{5.9}{\cancel{5.3(iii)}}$). Hence by 5.8(ii) $\sum_n |g_n| \in \mathcal{L}$. So $\sum_n |g_n|(x)$ is finite a.e. (Ex . . . .). Hence $\sum_n g_n(x)$ is (absolutely) convergent a.e. . Let g be as in (i) above. Now $g_n = g_n^+ - g_n^-$ and $g_n^{\pm} \geq 0$,(†) and $|g_n| = g_n^+ + g_n^- \geq g_n^{\pm}$. So $\int_{\mathbb{R}} g_n^{\pm} \leq \int_{\mathbb{R}} |g_n|$ and hence $\sum_n \int_{\mathbb{R}} g_n^{\pm} \leq \sum_n \int_{\mathbb{R}} |g_n| < \infty$. Thus $\sum g_n^{\pm} \in \mathcal{L}$ by 5.8(ii) and $\int_{\mathbb{R}} \sum_n g_n^{\pm} = \sum_n \int_{\mathbb{R}} g_n^{\pm}$.

For almost every x
$$g(x) = \lim_m \sum_1^m \{g_n^+(x) - g_n^-(x)\}$$
$$= \sum_1^\infty g_n^+(x) - \sum_1^\infty g_n^-(x)$$

and $\sum_n g_n^+$ and $\sum_n g_n^-$ are integrable. This shows that g is

(†) read the argument either with + or with − exclusively.

equal (a.e.) to the difference of two integrable functions, which is again an integrable function (5.5(i)). Apply 5.1 to conclude $g \in \mathcal{L}$. Now, (5.1 again),

$$\int_{\mathbb{R}} g = \int_{\mathbb{R}} \left(\sum_n g_n^+ - \sum_n g_n^- \right)$$

$$= \int_{\mathbb{R}} \sum_n g_n^+ - \int_{\mathbb{R}} \sum_n g_n^- \qquad \text{integral is linear}$$

$$= \sum_n \int_{\mathbb{R}} g_n^+ - \sum_n \int_{\mathbb{R}} g_n^- \qquad \text{by 5.8(ii) i.e. Monotone Convergence}$$

$$= \sum_n \int_{\mathbb{R}} g_n^+ - \int_{\mathbb{R}} g_n^- \qquad \text{(series converge abs')}$$

$$= \sum_n \int_{\mathbb{R}} g_n^+ - g_n^- \qquad \text{integral is linear}$$

$$= \sum_n \int_{\mathbb{R}} g_n \qquad g_n \equiv g_n^+ - g_n^-$$

$\square$.