2: Outer Sets and Inner Sets

We are now going to extend μ to a wider class of sets called "outer" sets. (The reason for this terminology will be clearer later.) From outer sets we obtain "inner" sets; the notions are "dual" to one another.

2.1 Definition

An outer set is a set which is the union of a sequence of basic sets.

2.2 Corollary

(i) If E is an outer set $\exists\ E_n \in \mathcal{B}$ $n = 1, 2, 3, \ldots$ such that $\bigcup_n E_n = E$. Let $F_n = \bigcup_{r=1}^{n} E_n$ then $F_n \in \mathcal{B}$, $F_n \subseteq F_{n+1}$ and $\bigcup_n^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n = E$. So any outer set is the union of an increasing sequence of basic sets.

(ii) Any basic set is an outer set.

(iii) by (i) any outer set E is the union of an increasing sequence of basic sets, (F_n) say. Let $G_n = F_{n+1} \setminus F_n$ $n \geq 2$ and $G_1 = F_1$. Then $\bigcup_n G_n = \bigcup_n F_n = E$ and E is the union of a sequence of disjoint sets $G_n \in \mathcal{B}$. Now each G_n is a union of disjoint intervals. Hence E is the union of a sequence of disjoint intervals.

(iv) Any set which is a countable union of intervals is an outer set

Hand out about extended real no's

2.3 Lemma

(i) Let (E_n) be an increasing sequence of basic sets and $E \in \mathcal{B}$ the union of the E_n's. Then $\lim_n \mu E_n = \mu E$.

(ii) Let (E_n) be a decreasing sequence of basic sets and $E = \bigcap_n E_n \in \mathcal{B}$. Then $\lim_n \mu E_n = \mu E$.

Proof

(i) Define $F_n = E_n \setminus E_{n-1}$ $\quad n \geq 2$ $\quad F_1 = E_1$ then $F_n \in \mathcal{B}$ and $F_n \cap F_m = \emptyset$ $\quad m \neq n$. Clearly (?) $\bigcup F_n = \bigcup E_n = E$ hence by 1·5 $\sum_{n=1}^{\infty} \mu F_n = \mu E$. But $\sum_{n=1}^{\infty} \mu F_n = \lim_{m \to \infty} \sum_{n=1}^{m} \mu F_n$

$= \lim_{m \to \infty} \mu\left(\bigcup_{n=1}^{m} F_n\right) = \lim_{m \to \infty} \mu(E_m)$ because $\bigcup_{n=1}^{m} F_n = E_m$.

(ii) Let $G_n = E_1 \setminus E_n$, as E_n's decrease then G_n's increase

now $\bigcup_n G_n = \bigcup_n (E_1 \setminus E_n) = E_1 \setminus \bigcap_n E_n = E_1 \setminus E$ by apply de Morgans laws (in the "universe" E_1). Hence,

(*) —— $\lim_{n \to \infty} \mu G_n = \mu(E_1 \setminus E) = \mu E_1 - \mu E$ $\quad$ ($E_1 = E_1\setminus E \cup E$ & μ is additive)

also $\lim_n \mu G_n = \lim_n \mu(E_1 \setminus E_n) = \lim_n (\mu E_1 - \mu E_n)$ and as (μE_n) is a decreasing sequence in $\mathbb{R}^+$, it converges thus enabling us to write

$$\lim_n \mu G_n = \mu E_1 - \lim_n \mu E_n = \mu E_1 - \mu E \quad \text{by } (*)$$

$\square$

Notation An increasing sequence of sets E_n with union E will be denoted $E_n \nearrow E$. Similarly $E_n \searrow E$ indicates a decreasing sequence of sets with intersection E.

4 Proposition

Let E be an outer set and suppose $E = \bigcup_n A_n = \bigcup_n B_n$ where $A_n \in \mathcal{B} \ni B_n$ $\forall n$ and $A_n \cap A_m = \emptyset = B_n \cap B_m$ whenever $n \neq m$. Then $\sum_{n=1}^{\infty} \mu A_n = \sum_{n=1}^{\infty} \mu B_n$.

Proof

$\forall m \in \mathbb{N}$ $\quad \bigcup_{n=1}^{m} A_n \subseteq \bigcup_n B_n = E$, since $\bigcup_{n=1}^{m} A_n \in \mathcal{B}$ we can apply 1·6(i)

to conclude $\quad \mu\left(\bigcup_{n=1}^{m} A_n\right) = \sum_{n=1}^{m} \mu A_n \leq \sum_{n=1}^{\infty} \mu B_n \quad$ hence $\sum_{n=1}^{\infty} \mu A_n \leq \sum_{n=1}^{\infty} \mu B_n$

By interchanging A_n and B_n in the argument above we obtain the reverse inequality $\sum_{n=1}^{\infty} \mu B_n \leq \sum_{n=1}^{\infty} \mu A_n$. If one of $\sum \mu A_n$ and $\sum_n \mu B_n$ is finite these relations show that the other is too and that they are equal. If one is infinite these relations show that the other is too, thus they are equal. □

The point of proving this proposition is revealed in

2·5 Definition.

Let E be an outer set and suppose $E = \bigcup_n E_n$, $E_n \in \mathcal{B}$. then writing $F_n = \bigcup_{k=1}^{n} E_k$ and $G_n = F_{n+1} \setminus F_n$, $G_1 = F_1$, as in 2·2 we define the measure of the outerset E by

$$\mu E = \lim_n \mu F_n = \sum_1^{\infty} \mu G_n \leq \infty$$

2·6 Corollary

The set function μ defined on outer sets is well-defined, and it is monotone, i.e. E, F outer, $E \subseteq F \Rightarrow \mu E \leq \mu F$. Moreover if $E = \bigcup_n J_n$ $J_n \in \mathcal{B}$ then $\mu E \leq \sum_n \mu J_n$

Proof By 2·4, Let $E = \bigcup_n E_n$, E_n's disjoint, $F = \bigcup_n F_n$ F_n's disjoint then $\mu E = \lim_m \sum_1^m \mu E_n$ but $\sum_1^m \mu E_n = \mu\left(\bigcup_1^m E_n\right) \leq \sum_1^{\infty} \mu F_n = \mu F$ by 1·6(i). So $\mu E \leq \mu F$.
With E and E_n as before, if $E = \bigcup_n J_n$ then $\forall m$ $\bigcup_{n=1}^{m} E_n \subseteq \bigcup_n J_n$ so $\sum_1^m \mu E_n \leq \sum_1^{\infty} \mu J_n$ by 1·6(i), passing to the limit gives the result.

2·7 Proposition

(i) An open subset of $\mathbb{R}$ is an outer set.

(ii) Let $\epsilon > 0$ and E be an outer set. There is an open set $F \supset E$ with $\mu F \leq \mu E + \epsilon$

Proof

(i) Recall that $G \subseteq \mathbb{R}$ is open $\iff \forall x \in G \ \exists \epsilon_x > 0 : (x-\epsilon_x, x+\epsilon_x) \subseteq G$. Let $x \in G$ define $L_x = \{y \in G : (y,x) \subseteq G,\ y < x\}$ and $R_x = \{y \in G : (x,y) \subseteq G\ \ x < y\}$. Since G is open L_x and R_x are non empty. Let $\ell_x = \inf L_x$ and $r_x = \sup R_x$.

We show that $(\ell_x, r_x) \subseteq G$. Let $z \in (\ell_x, r_x)$ and suppose first that $\ell_x < z < x$. Then $\exists y \in L_x : \ell_x < y < z$. But $y \in L_x$ so $(y,x) \subseteq G$, but $z \in (y,x)$ so $z \in G$. If however $z > x$ then $\exists y' \in R_x : x < z < y' < r_x$. But $y' \in R_x \Rightarrow (x,y') \subseteq G$ since $z \in (x,y')$ this shows $z \in G$. Finally we note that $x \in G$ by hypothesis so all points of (ℓ_x, r_x) lie in G. Let $x_1, x_2 \in G$ then $(\ell_{x_1}, r_{x_1}) \cap (\ell_{x_2}, r_{x_2})$ is either empty or non empty.

If the intersection is non empty, say $t \in (\ell_{x_1}, r_{x_1}) \cap (\ell_{x_2}, r_{x_2})$ then $(\ell_{x_i}, r_{x_i}) \subseteq (\ell_t, r_t)$ for $i = 1,2$ (a picture helps)

But then $x_i \in (\ell_t, r_t)$ and hence $\ell_{x_i} \leq \ell_t$ and $r_{x_i} \geq r_t$

but $(\ell_{x_i}, r_{x_i}) \subseteq (\ell_t, r_t)$ gives $\ell_{x_i} \geq \ell_t$ and $r_{x_i} \leq r_t$

So $\ell_{x_i} = \ell_t$ & $r_{x_i} = r_t$ $i = 1,2$. [Make a remark about equivalence relation $x \sim y \iff (x,y) \subseteq G$]

Hence $(\ell_{x_1}, r_{x_1}) = (\ell_{x_2}, r_{x_2})$! . So given $x_1, x_2 \in G$ (ℓ_{x_1}, r_{x_1}) & (ℓ_{x_2}, r_{x_2}) are either disjoint or they coincide.

Recall that any open interval contains a rational number.

Now the family $\{(\ell_x, r_x) : x \in G\}$ must be countable because if there are uncountably many distinct intervals each would contain a rational number distinct from the rationals in every other interval. We would have to conclude that there are uncountably many rationals. So

$$G = \bigcup_{x \in G} (\ell_x, r_x) \quad \text{is a union of disjoint open intervals}$$

and hence it is an outer set.

(ii) Let $E = \bigcup_n I_n$ where (I_n) is a sequence of disjoint intervals.

Let $\epsilon > 0$ and, as in 1·6, define J_n to be an open interval such that $J_n \supseteq I_n$, $\mu J_n < \mu I_n + \epsilon/2^n$. Then $F = \bigcup_n J_n$ is an outer set, it is open and $F \supseteq E$. Finally by 2·6

$$\mu F \leqslant \sum_n \mu J_n < \sum_n (\mu E_n + \epsilon/2^n) = \sum \mu E_n + \epsilon = \mu E + \epsilon. \square$$

Remark : Reason for terms inner + outer

2·8 Definition

(i) An inner set is a set which is bounded and whose complement is an outer set. (Equivalently)

$$E \text{ is inner} \iff E \text{ is bd and } \mathbb{R}\setminus E \text{ is outer}$$

Since an inner set, E, is bounded there is an interval I, $E \subseteq I$.

(ii) We define the measure of an inner set E as follows. Suppose $E \subseteq I$, I an interval. Since $\mathbb{R}\setminus E$ is outer there is a sequence of basic sets F_n, $F_n \uparrow \mathbb{R}\setminus E$. So $\mathbb{R}\setminus E = \bigcup_n F_n$ hence $E = \bigcap_n \mathbb{R}\setminus F_n$, since $E \subseteq I$ we have $E = I \cap (\bigcap_n \mathbb{R}\setminus F_n)$ $= \bigcap_n I\setminus F_n$. Now $(I\setminus F_n)$ is a decreasing sequence of basic sets and hence $\lim_n \mu(I\setminus F_n)$ exists, we define

$$\mu E = \lim_n \mu(I\setminus F_n)$$

Now one can object to this definition on two grounds. First the interval I is certainly not unique. Second the sequence of basic sets (F_n) is not unique either. We have to show that μE is well defined. Suppose first that J is an interval with $E \subseteq J$ let I and (F_n) be as above let $\mu_I(E) = \lim_n \mu(I\setminus F_n)$ and $\mu_J(E) = \lim_n \mu(J\setminus F_n)$ we want to show that $\mu_I(E) = \mu_J(E)$. A moments thought will convince you that it is enough to prove the following.

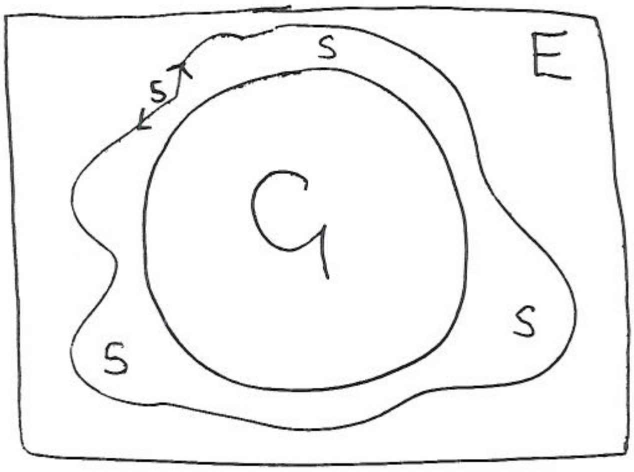
Let $E_n \in \mathcal{B}$ $E_n \downarrow G$ (G not necessarily a basic set) suppose also $F_n \in \mathcal{B}$

and ~~E~~ $\downarrow G$. Then $\lim_n \mu E_n = \lim_n \mu F_n$. To prove this let $k \in \mathbb{N}$ be fixed. Consider $E_k \setminus E_n$, $n = 1, 2, \ldots$. Then as each E_n is a basic set and $E_n \downarrow$ we have $(E_k \setminus E_n)$ is an increasing sequence of basic sets. Now $\bigcup_n E_k \setminus E_n = E_k \setminus \bigcap_n E_n = E_k \setminus G$, so $\forall k$ $E_k \setminus G$ is an outer set. Now $\forall k$ $G \subseteq F_k$ hence $E_k \setminus F_k \subseteq E_k \setminus G$ and by 2·6 μ is monotone on outer sets so $\mu(E_k \setminus F_k) \leq \mu(E_k \setminus G)$. But by definition $\mu(E_k \setminus G) = \lim_n \mu(E_k \setminus E_n) = \mu E_k - \lim_n \mu E_n$. Hence $\lim_k \mu(E_k \setminus F_k) = 0$. Now since $E_k = E_k \setminus F_k \cup (E_k \cap F_k)$ then $\lim_k \mu E_k = \lim_k \mu(E_k \cap F_k)$. This last discussion applies equally well with (E_k) replaced by (F_k) hence $\lim_k \mu F_k = \lim_k \mu(E_k \cap F_k) = \lim_k \mu E_k$. □

This discussion actually shows that μE, for an inner set E, is independent of both the interval I and the sequence (F_n).

In the next section we will define "finitely measurable sets". The definition is very much like that of the Riemann integral of a function in terms of upper and lower sums. The basic idea is just this; a set will be finitely measurable if we can find an outer set containing it and an inner set inside it which have measures that are close to each other. Effectively this means we must define the measure of $E \setminus G$ when E is outer & G is inner, so a few Lemma's about these matters will preceed our discussion of finitely measurable sets.

<u>Picture</u>



Require that $\forall \epsilon > 0$

$\exists$ E outer, G inner

$G \subseteq S \subseteq E$ and $\mu(E \setminus G) < \epsilon$ whatever that means!