§13 : Real Hilbert Space

A quite useful result about Lebesgue measurable and Borel measurable functions is the following ; Suppose that ϕ is Borel measurable and f is Lebesgue measurable and $\phi(0) = 0$ then $\phi \circ f$ is Lebesgue measurable.

Now it is reasonable easy to show that $x \mapsto |x|^\alpha$ is a Borel measurable function and thus that $|f|^\alpha$ is measurable whenever f is. The point of all this is that it shows that the following definition has some hope of capturing some functions.

13·1 Definition

(i) Let $f \in \mathcal{M}(\mathbb{R})$ (the measurable functions on $\mathbb{R}$) be such that $|f|^2 \in \mathcal{L}$, the f is said to be in the class $\mathcal{L}^2(\mathbb{R})$, we write $f \in \mathcal{L}^2(\mathbb{R})$

(ii) Let $E \subseteq \mathbb{R}$ be a measurable set (e.g. $[0,1]$) we say that a measurable function f is in the class $\mathcal{L}^2(E)$ if $|f|^2 \chi_E \in \mathcal{L}$

Exercise Show that $\mathcal{L}^2(\mathbb{R}) \cap \mathcal{L} \neq \emptyset$ but that $\mathcal{L}^2(\mathbb{R}) \neq \mathcal{L}$.

13·2 Proposition

Let $f, g \in \mathcal{L}^2(\mathbb{R})$ then $f \cdot g \in \mathcal{L}$.

Proof

By $f \cdot g \in \mathcal{M}(\mathbb{R})$ and since

$$|f(x)g(x)| \leq \frac{f^2(x) + g^2(x)}{2} \qquad \begin{pmatrix}(a+b)^2 \geq 0 \\ (a-b)^2 \geq 0\end{pmatrix}$$

then by Remark following 7:10 $fg \in \mathcal{L}$. $\square$

Exercise Show that $\mathcal{L}^2(\mathbb{R})$ is a linear space.

13·3 Definition

ⓐ Let $f, g \in \mathcal{L}^2(\mathbb{R})$ then $\langle f, g\rangle \stackrel{\text{def}}{=} \int_{\mathbb{R}} fg$.

ⓑ $\|f\|_2 \stackrel{\text{def}}{=} \langle f, f\rangle^{1/2} \equiv \left(\int_{\mathbb{R}} |f|^2\right)^{1/2}$.

13·4 Proposition

(i) $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{L}^2(\mathbb{R})$. (Go and look up Def^n from 1P2.)

(ii) $\langle f, g\rangle = \langle g, f\rangle$ $\quad f, g \in \mathcal{L}^2(\mathbb{R})$

(iii) $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$ $\quad f, g, h \in \mathcal{L}^2(\mathbb{R})$

(iv) $\langle cf, g\rangle = c\langle f, g\rangle$ $\quad f, g \in \mathcal{L}^2(\mathbb{R}), c \in$

(v) $|\langle f, g\rangle| \leq \|f\|_2 \|g\|_2$

(vi) $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$

Proof (i) – (iv) are easily verified and (ii), (iii), (iv) are just elaborations of (i).

(v) Consider

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x)\, g(y) - g(x)\, f(y)|^2 \, dy \right) dx \geq 0$$

(vi) $\|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2 + 2\langle f, g\rangle$ now use (

and take square roots. $\square$

13·5 Proposition

Let $d(f,g) = \|f-g\|_2$ for $f, g \in \mathcal{L}^2(\mathbb{R})$.

Then $d(\cdot,\cdot)$ is a semi-metric i.e. "a "metric" whic fails to be a metric only in that one can have $d(f,g) = 0$ and $f \neq g$."

Proof

ⓐ $d(f,g) = d(g,f)$ ⓑ $d(f,f) = 0$ ⓒ

$d(f,g) \leqslant d(f,h) + d(h,g)$

But if $f = g$ a.e. $f - g = 0$ a.e. so that $\|f-g\|_2 = \left(\int_{\mathbb{R}} (f-g)^2\right)^{1/2} = 0$. (We could easily arrange $g \neq f$.). $\square$

This proposition shows us how to make a metric space from $\mathcal{L}^2(\mathbb{R})$. We must do just wh we did when we constructed $L^1(\mathbb{R})$ from $\mathcal{L}(\mathbb{R})$, ie we identify functions equal almost everywhere. Formally we define $f \sim g \iff f = g$ a.e. . This breaks $\mathcal{L}^2(\mathbb{R})$ into equivalence classes $[f]$, By defining the operations of addition and scalar multiplication via representatives we get a linear space. By defining $\|[f]\|_2 = \|f\|_2$ we get a norm which gives a metric $\rho([f],[g]) = \|f-g\|$. Once again we blur the distinction between f and its equivalence class $[f]$, i.e. we think of the class $[f]$ as the function f. We call $\{[f] : f \in \mathcal{L}^2(\mathbb{R})\}$, with the operations defined above and the norm given above, $L^2(\mathbb{R})$. With $([f],[g]) \stackrel{def}{=} \langle f, g\rangle$ we get a positive definite inner product $(\cdot\,,\cdot)$ on $L^2(\mathbb{R})$

<u>The</u> result we will prove in this section is that $L^2(\mathbb{R})$ is complete in the metric given by $\|[f]-[g]\|_2$. The proof is similar to that for $L^1(\mathbb{R})$. Once again we think of classes $[f]$ as the function f.

<u>13.6 Lemma</u>

Let $(g_n) \subset L^2(\mathbb{R})$ and $\sum_{n=1}^{\infty} \|g_n\|_2 < \infty$.

Then $\sum_n g_n(x) = g(x)$ exists and defines an element of $L^2(\mathbb{R})$.

<u>Proof</u> Let $M = \sum_{n=1}^{\infty} \|g_n\|_2$. Think of g_n's as functions.

$$\forall n \quad \Big\| \sum_{k=1}^{n} |g_k| \Big\|_2 \leq \sum_{k=1}^{n} \|g_k\|_2 \leq M$$

Hence $\displaystyle\int_{\mathbb{R}}\Big(\sum_{k=1}^{n}|g_k(x)|\Big)^2 dx = \Big\|\sum_{k=1}^{n}|g_k|\Big\|_2^2 \leqslant M^2$

Let $f_n = \Big(\sum_{k=1}^{n}|g_k|\Big)^2$ then $f_n \in \mathcal{L}$ and $f_n \uparrow$

and $\sup_n \int_{\mathbb{R}} f_n < \infty$. By Monotone convergence $\exists f \in$

$\therefore$ $f_n \uparrow f$ a.e. and $\lim_n \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$.

Hence $\sum_{k=1}^{\infty}|g_k(x)|$ converges a.e. and thus $\sum_{k=1}^{\infty} g_k(x)$ does too. Let $g(x) = \sum_{k=1}^{\infty} g_k(x)$ for those x for which the series converge. (zero otherwise?) Now consider

$\Big|\sum_{k=1}^{n} g_n(x)\Big|^2 = H_n(x)$. Then $H_n \in \mathcal{L}$ and $H_n \to |g|^2$ a.e., also $H_n(x) \leqslant f_n(x) \leqslant f(x)$ a.e. thus by Dominated convergence $|g|^2 \in \mathcal{L}$ and $\int_{\mathbb{R}}|g|^2 = \lim_n \int_{\mathbb{R}} H_n$.

Since we know that g is measurable (it is the a.e. limit of the sequence $\sum_{k=1}^{n} g_k$) we have $g \in \mathcal{L}^2(\mathbb{R})$. We arrive at the statement of the lemma by considering classes of functions equal a.e. . $\square$

13·7 Theorem (Riesz-Fischer)

Let (f_n) be a sequence in $L^2(\mathbb{R})$ which is Cauchy in norm, i.e., $\forall \epsilon > 0 \ \exists N \in \mathbb{N} : m, n \geq N : \|f_n - f_m\|_2 < \epsilon$

Then there is $f \in L^2(\mathbb{R})$: $\|f_n - f\|_2 \to 0$ as $n \to \infty$.

So that $L^2(\mathbb{R})$ is a complete normed space.

Proof

Using the Cauchy condition we may construct a rapidly Cauchy subsequence (f_{n_k}) just as in 6·2, i.e. we get positive integers $n_1 < n_2 < \dots$ with $\|f_m - f_{n_k}\|_2 < 1/2^k$ for $m \geq n(k)$.

Let $g_1 = f_{n_1}$, $g_k = f_{n_k} - f_{n_{k-1}}$, $k \geq 2$. Then

$$\sum_{k=1}^{\infty} \|g_k\|_2 = \sum_{k=2}^{\infty} \|f_{n_k} - f_{n_{k-1}}\|_2 + \|f_{n_1}\|_2 < \|f_{n_1}\|_2 + 1$$ anc

each $g_k \in L^2(\mathbb{R})$. By Lemma 13·6 $\sum_{k=1}^{\infty} g_k$ converge a.e. to a (function) $f \in \mathcal{L}^2(\mathbb{R})$. Note that

$$\|f - f_m\|_2 \leq \|f_m - f_{n_k}\|_2 + \|f_{n_k} - f\|_2 ,$$

we know $\|f_m - f_{n_k}\|_2 < 1/2^k$ and so it remains to estimate $\|f_{n_k} - f\|_2$. Now

$$f - f_{n_k} = \sum_{r=k+1}^{\infty} (f_{n_r} - f_{n_{r-1}}) \quad \& \quad \sum_{r=k+1}^{\infty} \|f_{n_r} - f_{n_{r-1}}\|_2$$ Conv

So $$\|f - f_{n_k}\|_2 \leq \sum_{r=k+1}^{\infty} \|f_{n_r} - f_{n_{r-1}}\|_2 < \sum_{r=k+1}^{\infty} 1/2^{r-1} < 1/2^{k-1} .$$

Hence $\| f - f_m \|_2 < \frac{3}{2^k}$ if $m \geq n_k$ and as $n_k \to \infty$ as $k \to \infty$ this shows $\| f - f_m \|_2 \to 0$ as $m \to \infty$.

$\square$

Complex Hilbert Space

Let $f : \mathbb{R} \to \mathbb{C}$ and let $\overline{f(x)}$ denote the complex cngugate of $f(x)$. The functions

$u = \frac{f + \overline{f}}{2}$ and $\frac{f - \overline{f}}{2i} = v$ are, respectively the real and imaginary arts of f. We say that f is measurable if both u and v are, integrable over a measurable set E, if both u and v are,

One can check that $\mathcal{L}^2_{\mathbb{C}} = \{ f : \mathbb{R} \to \mathbb{C} \ : \ f$ is measurable and $|f|^2 \in \mathcal{L}\}$ is a linear space with the usual operations. Taking equivalenc classes(†) yeilds $L^2_{\mathbb{C}}$. With the inner product given by,

$$(f, g) = \int_{\mathbb{R}} f \overline{g}$$

$L^2_{\mathbb{C}}$ is a complete inner product space.

(†) $f \sim g \Leftrightarrow f = g$ a.e.