10: Abstract Measure Theory

Despite the fact that our development of Lebesgue measure used notions such as the lengths of intervals, the order structure on $\mathbb{R}$, the Hiene-Borel theorem. Notion which may at first sight seem essential to the develo-ment and peculiar to $\mathbb{R}$, it is possible to abstract the essential features of our discussion of measure and integration and cast it in a very general context. This abstraction is valuable because it makes us realise what features are essential in our treatment of measure and integration as opposed to those that are incidental.

The first thing that we abstract is the idea of a basic set, this gives a system called a ring.

10·1 Definition

Let Ω be a set. A non empty collection $\mathcal{J}$ of subsets of Ω is called a ring iff

(i) $\emptyset \in \mathcal{J}$,

(ii) $A, B \in \mathcal{J} \Rightarrow A \cup B \in \mathcal{J}$

(iii) $A, B \in \mathcal{J} \Rightarrow B \setminus A \in \mathcal{J}$

Remark (i) actually follows from (iii). A special case ~~Yes, $\mathcal{B}$ is not a field. But it fails to be one because $\mathbb{R} \in \mathcal{B}$~~ of a ring is a ring with $\Omega \in \mathcal{J}$. Such rings are called fields or algebras. Note that $\mathcal{B}$ is not a field, but is a ring.

Recall that we took our basic sets and looked at first outer sets, formed from increasing sequences of basic sets, then inner sets, formed from decreasing sequences of basic sets. We extended the measure μ from $\mathcal{B}$ to the inner and outer sets (allowing $\mu E = \infty$). Finally we selected the finitely measurable sets and extended the measure to these. Later on we identified measurable sets these included the finitely measurable sets and the inner and outer sets (of finite or infinite measure). It turns out that the measurable sets are the next element of the abstraction because they form a system called a σ-ring

10:2 Definition

A σ-ring is a ring which is closed under countable unions. Thus if $E_n \in \mathcal{J}$ $n=1,2,\dots$ then $\bigcup_n E_n \in \mathcal{J}$ (to use the notation of 10·1).

Remark A σ-algebra is a σ-ring with $\Omega \in \mathcal{J}$.

One can prove the following

Theorem 10·3 (Halmos: §5: Thm A + remarks preceeding Thm D)

Let $\mathcal{J}$ be any non empty collection of subsets of Ω. Then there is a smallest (with respect to $\subseteq$) ring | algebra containing $\mathcal{J}$ and a smallest σ-ring | σ-algebra containing $\mathcal{J}$, we denote these by $R(\mathcal{J})$ and $\sigma(\mathcal{J})$.
(We call $\sigma(\mathcal{J})$ the σ-ring generated by $\mathcal{J}$.)

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A σ-ring is a ring which is closed under countable unions. Thus if $E_n \in \mathcal{F}$ $n=1,2,\ldots$ then $\bigcup_n E_n \in \mathcal{F}$ (to use the notation of 10.1).

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The reason that rings and σ-rings are of some importance becomes clear when we consider a <u>measure</u> on a ring. You could probably guess the following

10:4 Definition.

Let Ω be a set, $\mathcal{F}$ a ring of subsets of Ω. A function $\mu : \mathcal{F} \to \overline{\mathbb{R}}$ is called a <u>measure</u> iff

(i) $\mu(\emptyset) = 0$

(ii) If $(E_n) \subset \mathcal{F}$, $E_n \cap E_m = \emptyset$, $m \neq n$, and $\bigcup_n E_n \in \mathcal{F}$ then $\mu\left(\bigcup_n E_n\right) = \sum_n \mu E_n$

<u>Remark.</u> 1·6 (ii) shows that "Lebesgue measure" is a measure on the ring $\mathcal{B}$.

The point is that one can extend a measure from a ring $\mathcal{F}$ to $\sigma(\mathcal{F})$, the σ-ring generated by $\mathcal{F}$, analogous to what we have done in extending Lebesgue measure from $\mathcal{B}$ to the measurable sets. Our 'integrable' sets are just those to which the measure μ extends with a finite value.

I will not give you the details of the extension of a measure, μ, on a ring $\mathcal{F}$ to the σ-ring, $\sigma(\mathcal{F})$. But a summary may entice you to read further.

Let $\mathcal{F}$ be a ring and $H(\mathcal{F})$ the class of all sets which can be covered by countably many sets in $\mathcal{F}$. $H(\mathcal{F})$ is a σ-ring. Define an <u>outer measure</u> on $H(\mathcal{F})$ by

$$\mu^* E = \inf \left\{ \sum_n \mu E_n : E_n \in \mathcal{F} (\forall n), E \subseteq \bigcup_n E_n \right\} \quad \text{where } \mu \text{ is a measure on}$$

μ^* is $\mathbb{R}$ valued, non negative, monotone, countably subaddi[tive]
and $\mu^*(\emptyset) = 0$. But μ^* may not be additive on $H(\mathcal{F})$.
In order to select those sets where μ^* is additive one (thinks and fiddles for a long time and then) defines;

A set $E \in H(\mathcal{F})$ is μ^*-measurable if for every $A \in H(\mathcal{F})$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E').$$

Let $\overline{\mathcal{F}}$ be the set of μ^*-measurable sets. $\overline{\mathcal{F}}$ is a σ-ring, $\mathcal{F} \subseteq \overline{\mathcal{F}}$ and μ^* is a measure on $\overline{\mathcal{F}}$ which agrees with μ on $\mathcal{F}$. Since $\sigma(\mathcal{F})$ is the smallest σ-ring containing $\mathcal{F}$ we must have $\sigma(\mathcal{F}) \subseteq \overline{\mathcal{F}}$ and $\mu^* \restriction \sigma(\mathcal{F})$ is a measure extending $\mu \restriction \mathcal{F}$. So we have done a little better than extending μ from $\mathcal{F}$ to $\sigma(\mathcal{F})$!
How does this look in our concrete set up on the real line?. If we start with $\mathcal{B}$ and Lebesgue measure μ on $\mathcal{B}$ then $\overline{\mathcal{B}}$ is precisely the class of Lebesgue measurable sets.
The σ-ring generated by $\mathcal{B}$, $\sigma(\mathcal{B})$, is called the Borel σ-field (yes <u>field</u> because this particular ring generates a σ-ring which is actually a σ-field).
A natural question is; What's the difference between a Borel set and a Lebesgue measurable set? The answer is, roughly speaking a null set, viz;

10:5 Theorem

(a) Let μ be a measure on a σ-ring S. Let $\overline{\overline{S}}$ be the collection of all sets of the form $E \cup N$ where N is any subset of a set of measure zero in S. $\overline{\overline{S}}$ is a σ-ring and μ extends to $\overline{\overline{S}}$ via $\overline{\overline{\mu}}(E \cup N) = \mu E$. We call $\overline{\overline{S}}$ the <u>completion</u> of S (with respect to μ).

(b) Let μ be Lebesgue measure on $\mathcal{B}$. Then $\overline{\mathcal{B}} = \overline{\overline{\sigma(\mathcal{B})}}$ and $\overline{\overline{\mu}} = \mu^*$. That is, the Lebesgue sets are just the completion of the Borel sets. You just have to chuck in all subsets of Borel sets with zero measure.

As you can see, Measure Theory starts to get quite technical and tricky which is a good point at which to leave and to consider,

11: Abstract Integration Theory.

Let Ω be a set, Σ a σ-ring of subsets of Ω and μ a measure on Σ. Let us suppose also that Ω is the union of a sequence in Σ, (E_n) say, with $\forall n \;\; \mu E_n < \infty$. The triple (Ω, Σ, μ) is called a (σ-finite) measure space. If $\mu\Omega < \infty$ (Ω, Σ, μ) is called a <u>finite</u> measure space and if $\mu\Omega = 1$ a <u>probability space</u>.

Examples

(i) $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+), \lambda)$

(ii) $([0,1], \mathcal{B}([0,1]), \lambda)$