Hahn-Banach Theorems for Mathematical Finance

You will need to know the definitions of real linear space, subspaces, linear functionals on real linear spaces and have a small stock of concrete examples of these to aid your intuition.

Preliminaries

Let L be a real linear space. For x and y in L, $x \neq y$, the line between x and y is the subset of L given by $\lambda x + (1-\lambda)y$, $\lambda \in [0,1]$. The line through x and y is given by $\lambda x + (1-\lambda)y$, $\lambda \in \mathbb{R}$. A non-empty subset, E, of L is flat if for each pair of distinct points, x, y in E, E contains the line through x and y.

Lemma 1

If E is a flat subset of L and $x \in E$ then $E - x$ is a subspace of L.

Pf

Let $\lambda \in \mathbb{R}$ and $e \in E$. Since the line through e and x lies in E then $\lambda e + (1-\lambda)x \in E$. Therefore $\lambda e + (1-\lambda)x - x = \lambda(e-x) \in E - x$. So $E-x$ is closed under scalar multiplication. Now suppose e_1, e_2 are in E and $e_1 \neq e_2$. We know that $\frac{e_1+e_2}{2} \in E$ and so $\frac{e_1+e_2}{2} - x \in E - x$. But by the first part, $E - x \ni 2\left(\frac{e_1+e_2}{2} - x\right) = e_1 + e_2 - 2x$ $= (e_1 - x) + (e_2 - x)$. So $E-x$ is closed under addition.

$\square$

A hyperplane, H, in L is a maximal flat subset of L; i.e. H is flat and if M is flat, $M \supseteq L$, then either $M = L$ or $M = H$. We restrict hyperplanes to be <u>proper</u> subsets of L.

<u>Lemma 2</u>

H is a hyperplane in L if and only if for each $x \in H$ $H-x$ is a maximal proper subspace of L.

<u>Pf</u>

By the previous Lemma, $H-x$ is a subspace of L, $H-x$ is a proper subspace because otherwise H is <u>not</u> a proper subset of L. If M is a subspace of L then M is flat, obviously, so given m_1, m_2 in M and $\mu \in \mathbb{R}$, $\mu m_1 + (1-\mu) m_2 \in M$. Now,

$$\mu(m_1 + x) + (1-\mu)(m_2 + x) = \mu m_1 + (1-\mu) m_2 + x.$$

This shows that $M+x$ is flat. So, if $M \supseteq H-x$ then $M+x \supseteq H$. Since H is a hyperplane, $M+x$ is either H itself or it is L, so M is either $H-x$ or it is $L-x \equiv L$. This shows that $H-x$ is <u>maximal</u>. If $H-x$ is a maximal proper subspace then $(H-x)+x$ is flat and using the first part, proper and maximal.

<u>Corollary 3</u>

H is a hyperplane if and only if $H = E + x$ where E is a maximal proper subspace.

<u>Pf</u>

For $x \in H$, $H-x$ is a maximal proper subspace. If H is the translate of a maximal proper subspace, E, then $x \in H$, $E + x$ is flat, proper and maximal — see proof of Lemma 2.

$\square$

Let f be a linear functional on L, i.e. $f \in L^{\#}$.

If $f \neq 0$ then $\ker f = \{\ell \in L : f(\ell) = 0\}$ is a proper subspace of L. Let $y \in L$ be such that $f(y) \neq 0$.

Lemma 4

$$L = \ker f \oplus \langle y \rangle$$

Pf

Certainly $\ker f + \langle y \rangle$ is a subspace of L. Now suppose that $k + \lambda y \in \ker f + \langle y \rangle$ is zero. This states that $k + \lambda y = 0 \Leftrightarrow k = -\lambda y \Rightarrow y = \left(\frac{-1}{\lambda}\right) k$ in case $\lambda \neq 0$. But this means $f(y) = 0$, ※. So we must have $\lambda = 0$ and therefore $k = 0$. In other words the sum, $\ker f + \langle y \rangle$ is direct. Given $\ell \in L$ with $f(\ell) \neq 0$ there is an element of $\ker f \oplus \langle y \rangle$, $k + \lambda y$ say, with $f(k+\lambda y) = f(\ell)$. So $\ell - (k+\lambda y) \in \ker f$. That is,

$$\ell = k_1 + (k + \lambda y) \quad , k_1 \in \ker f$$

so $$\ell = (k_1 + k) + \lambda y \in \ker f \oplus \langle y \rangle.$$

If $f(\ell) = 0$, $\ell \in \ker f$. So $L = \ker f \oplus \langle y \rangle$. □

Corollary 5

If $f \in L^{\#}$ and $f \neq 0$ then $\ker f$ is a maximal proper subspace of L and for $\lambda \neq 0$, $\{\ell : f(\ell) = \lambda\}$ is a hyperplane in L.

Pf

Aliter: $\ker f$ has co-dimension equal to 1.

Let M be a subspace containing $\ker f$. Either $M = \ker f$ or

M contains an element, y, with $f(y) \neq 0$. As we have seen, $L = \ker f \oplus \langle y \rangle \subseteq M$. So $M = L$, and $\ker f$ is maximal. Let $H = \{x : f(x) = \lambda\}$ and $\ell \in H$, we know H is non-empty, clearly $H - \ell \subseteq \ker f$. If $k \in \ker f$ then $k + \ell \in H$, and $k \in H - \ell$, i.e., $\ker f \subseteq H - \ell$. So $H = \ker f + \ell$ and by Corollary 3 it is a hyperplane.

__Corollary 6__

Let H be a hyperplane in L. There is a non-zero $f \in L^{\#}$ with a $\lambda \neq 0$(†) in $\mathbb{R}$ such that

$$H = \{\ell \in L : f(\ell) = \lambda\}.$$

__Pf__ From Corollary 3, $H = E + x$, where E is a maximal proper subspace. If $x \in E$ we take $y \in L \setminus E$ and define f by $f \restriction E \equiv 0$ and $f(\mu x) = \mu$. So in this case $E = \{\ell : f(\ell) = 0\}$. If $x \notin E$ then, again, $f \restriction E \equiv 0$ and set $f(x) = \lambda$ so that f is defined on $L = \ker f \oplus \langle x \rangle$ by $f(k + \mu x) = \mu\lambda$. Clearly $H \subseteq \{\ell : f(\ell) = \lambda\}$ whereas if $f(\ell) = \lambda$ then for each $k \in E$, $f(\ell - (k + x)) = 0$, so $\ell - (k + x) = k_1$, for some $k_1 \in E$. This means $\ell = (k_1 + k) + x \in E + x = H$. So $H = \{\ell : f(\ell) = \lambda\}$.

__Corollary 7__

Let H be a hyperplane.(††) Then,

$$L = \bigcup_{t \in \mathbb{R}} tH$$

__Pf__ By Corollary 6, $H \equiv H_\lambda = \{\ell \in L : f(\ell) = \lambda\}$ for $f \in L^{\#}$ and $\lambda \neq 0$ (we discuss the $\lambda = 0$ case separately). Now,

(†) When H is a subspace, see the proof of Corollary 6 for a discussion of this case.
(††) This doesn't work if H is a subspace.

$$H_\lambda = \{ l \in L : f(l) = \lambda \} = \{ l \in L : f(\tfrac{l}{\lambda}) = 1 \}$$

$$= \{ \lambda l : f(l) = 1 \} = \lambda H_1 .$$

So for $t \in \mathbb{R}$, $tH \equiv tH_\lambda = t\lambda H_1$. So if $l \in L$ and $f(l) = \mu \in \mathbb{R}$ then $l \in \frac{1}{\mu} H_1$, i.e. $l \in tH$ where $t = \frac{\mu}{\lambda}$.

If $\lambda = 0$ this result 'fails' because multiplying by 't' doesn't 'move' the hyperplane around. Notice that when $\lambda \neq 0$ and $H = \{ l : f(l) = \lambda \}$ then $tH = \{ l : f(l) = t\lambda \}, t \neq 0$, so tH is also a hyperplane.

$\square$

Recall that a function, $p : L \to \mathbb{R}$ is <u>sublinear</u> iff $\forall \lambda > 0$, $x, y \in L$ we have,

$$p(x+y) \leq p(x) + p(y)$$

$$p(\lambda x) = \lambda p(x).$$

Sublinear functionals are used to describe prices when one has distinct buying and selling prices. Roughly speaking, $p(x)$ is the buy price of x and $-p(-x)$ is what you get when you (short) sell it. Notice that;

(i) $p(0) = p(x - x) \overset{\leq}{=} p(x) + p(-x) = p(x) - (-p(-x))$

So $p(x) \geq -p(-x)$ and the buy price exceeds the sell price.

(ii) $\forall x, p(2x) = 2p(x)$ so $p(0) = 2p(0) \Rightarrow p(0) = 0.$

Sublinear functionals are a feature of the Hahn-Banach theorem. This result concerns itself with the existence of extensions of linear functionals, defined on a proper subspace of L, to the whole of L. It turns out that this result, and some of its kin, are key technologies in the theory of pricing of contingent claims and have a direct relationship to questions of arbitrage or the lack of it! In the sequel we give a presentation of some of these results and later(†) their use in Mathematical Finance. Naturally this won't be an exhaustive treatment. We start with the classic Hahn-Banach Theorem.

(†) Much later!

Let L be a linear space over $\mathbb{R}$ and L_o a subspace of L and $f \in L_o^{\#}$. Suppose also that there is a sublinear functional, p, defined on L such that $f(\ell) \leq p(\ell)$, $\ell \in L_o$. The Hahn-Banach theorem asserts that f has an extension to all of L which remains dominated by p.

Proof

Let L' be a subspace of L, $L_o \subseteq L'$, and f' a linear functional defined on L' which satisfies, $f'(\ell_o) = f(\ell_o)$, $\ell_o \in L_o$. We say f' extends f. Now let $\mathcal{F}$ denote the set of linear functionals which extend f and are dominated by p on their domain. Then $\mathcal{F}$ is non-empty because it contains f itself. The relation, $f_1 \leq f_2 \Leftrightarrow f_2$ extends f_1, partially orders $\mathcal{F}$. Let Υ be a totally ordered subset of $\mathcal{F}$. By defining $\hat{f}(\ell) = f'(\ell)$ on $\mathrm{dom}(f')$, for $f' \in \Upsilon$, we see that $\hat{f}$ is a least upper bound for Υ (and it lies in $\mathcal{F}$, of course). According to Zorns Lemma, $\mathcal{F}$ has a maximal element. Let f^o be this maximal element. If $\mathrm{dom}(f^o) = L$ we are finished. If $\mathrm{dom}(f^o) \neq L$ then there must be $h \in L$ outside of $\mathrm{dom}(f^o)$. We show that this cannot be so by constructing an extension to f^o which is dominated by p, and therefore contradicting the fact that f^o is maximal. So let us investigate what would be true under the assumption that,

(i) $\exists\, h \in L \setminus \mathrm{dom}(f^o)$,

(ii) $\exists\, f' \geq f^o$, $f' \neq f^o$,

(iii) $h \in \mathrm{dom}(f')$.

We will think about the subspace $\mathrm{dom}(f^o) + \langle h \rangle$.

On $\mathrm{dom}(f^o)$, f' agrees with f^o while on $\langle h \rangle$, being a linear functional, it has the form $F'(\lambda h) = \lambda c$ where $c = f'(h)$ of course. So on $\mathrm{dom}(F^\circ) + \langle h \rangle$ we have

$$F'(\ell + \lambda h) = F^\circ(\ell) + \lambda c.$$

Now in order for this to be dominated by p we require

$$F^\circ(\ell) + \lambda c \leq p(\ell + \lambda h)$$

for $\lambda \in \mathbb{R}$ and $\ell \in \mathrm{dom}(F^\circ)$. For $\lambda = 0$, this is true.
For $\lambda > 0$ we have, true iff

$$\begin{aligned} c &\leq \tfrac{1}{\lambda} p(\ell + \lambda h) - F^\circ(\tfrac{\ell}{\lambda}), \\ &= p(\tfrac{\ell}{\lambda} + h) - F^\circ(\tfrac{\ell}{\lambda}). \end{aligned}$$

For $\lambda < 0$ we have, true iff

$$\begin{aligned} c &\geq \tfrac{1}{\lambda} p(\ell + \lambda h) - F^\circ(\tfrac{\ell}{\lambda}), \\ &= -p(-\tfrac{\ell}{\lambda} - h) - F^\circ(\tfrac{\ell}{\lambda}). \end{aligned}$$

Now, as ℓ runs over the subspace $\mathrm{dom}(F^\circ)$, then $\frac{\ell}{\lambda}$ runs over all elements of $\mathrm{dom}(F^\circ)$ (†), so the first of our inequalities is asking that

$$c \leq \inf\{\, p(\ell + h) - F^\circ(\ell) : \ell \in \mathrm{dom}(F^\circ) \,\}$$

while the second inequality wants

$$c \geq \sup\{\, -p(-(\ell + h)) - F^\circ(\ell) : \ell \in \mathrm{dom}(F^\circ) \,\}.$$

(†) Whether λ be positive or negative

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Now clearly, because p is sub-additive then

$$-p(-(l+h)) - f^\circ(l) \leq p(l+h) - f^\circ(l)$$

but this is for a single $l \in \mathrm{dom}(f^\circ)$, what the inequality requires is

$$-p(-(l'+h)) - f^\circ(l') \leq p(l+h) - f^\circ(l)$$

for all choices of $l, l' \in \mathrm{dom}(f^\circ)$. But is this so? Observe,

$$f^\circ(l - l') \leq p(l - l') = p(l + h - h - l')$$

$$\leq p(l+h) + p(-(l'+h))$$

So $\quad -p(-(l'+h)) - f^\circ(l') \leq p(l+h) - f^\circ(l),$

for every $l, l' \in \mathrm{dom}(f^\circ)$. So, indeed,

$$-\infty < A = \sup\{-p(-(l'+h)) - f^\circ(l') : l' \in \mathrm{dom}(f^\circ)\}$$

$$\leq \inf\{p(l+h) - f^\circ(l) : l \in \mathrm{dom}(f^\circ)\} = B < \infty$$

And we can choose $c \in [A, B]$ to be sure our extension exists and is dominated by p. Enough.