$$\langle f, g \rangle = \langle Z, g \rangle \quad \text{for} \quad g \in \bigcup_n L^2(\Omega, \mathcal{F}_n, \mathbb{P}).$$

Of course $\bigcup_n L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ is not all of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ so we are not yet finished. One way of completing the proof is to observe that for sets $E \in \bigcup_n \mathcal{F}_n$ (which is a field), the function

$$E \longmapsto \langle f, I_E \rangle = \int_\Omega f I_E \, d\mathbb{P} = \int_\Omega Z I_E \, d\mathbb{P}$$

is a countably additive (measure) set function and so — recall our early assumptions of the course, extends to a unique countably additive measure on (all of) the σ-field generated by $\bigcup_n \mathcal{F}_n$, that is, on $\mathcal{F}$. So $\int_\Omega f I_E \, d\mathbb{P}$ and $\int_\Omega Z I_E \, d\mathbb{P}$ must agree on all of $\mathcal{F}$ and hence on all of $L^2(\Omega, \mathcal{F}, \mathbb{P})$

i.e., we have, $\langle f, g \rangle = \langle Z, g \rangle$, $g \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.

So $f = Z$.

There is a longer, but more elementary proof that $f = Z$, which we look at now.

The idea of the proof is this; The set of elements of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, $\{ \lim_n M_{\mathcal{F}_n}(f) : f \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \}$ is a closed subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$. We will show that I_E, for $E \in \mathcal{F}$, must lie in this 'closed subspace', hence that $L^2(\Omega\ \mathcal{F}\ \mathbb{P})$ is equal this subspace.

<u>Step 1</u> Recall that $\forall n$ $\|M_{\mathcal{F}_n}(f)\| \leqslant \|f\|$

and so if $P(f) = \lim_n M_{\mathcal{F}_n}(f)$ then

$\|P(f)\| \leqslant f$. So the "map" P (i.e. $f \longmapsto P(f)$) is linear (limits are linear) and continuous ($f_n \to f$ in $\|\cdot\|$ then $P(f_n) \to P(f)$ in $\|\cdot\|$). Now P has one more crucial property

$$P \circ P = P$$

(you might guess that P is the orthogonal projection onto $\{P(f) : f \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$ — you're right!)

To see this observe, $P^2(f) = P\left(\lim_n M_{\mathcal{F}_n}(f)\right) =$

$$= \lim_k M_{\mathcal{F}_k}\left(\lim_n M_{\mathcal{F}_n}(f)\right) = \lim_k \left(\lim_n M_{\mathcal{F}_k} \circ M_{\mathcal{F}_n}(f)\right)$$

For $M_{\mathcal{F}_k}$ is continuous $(M1(ii))$ $= \lim_k \left(\lim_n M_{\mathcal{F}_k}(f)\right)$

because $M_{\mathcal{F}_k} \circ M_{\mathcal{F}_n} = M_{\mathcal{F}_k}$ for $n \geqslant k$, (Lemma M2)

$$= \lim_k M_{\mathcal{F}_k}(f)$$

$$= P(f).$$

So if $P(f_n) \longrightarrow a$ then $P(P(f_n)) \longrightarrow P(a)$ for P

is continuous, but $P(P(f_n)) = P^2(f_n) = P(f_n)$ so $P(g) = g$, i.e. $\{P(f) : f \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$ is a closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

<u>Step 2</u>

Now let $G = \{E \in \mathcal{F} : P(I_E) = I_E\}$, i.e. all sets in $\mathcal{F}$ whose indicator functions lie in $\{P(f) : f \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$. Notice that if $E \in \mathcal{F}_n$ then $M_{\mathcal{F}_k}(I_E) = I_E$ for each $k \geq n$ (the nearest thing to I_E in $L^2(\Omega, \mathcal{F}_k, \mathbb{P})$ is itself!). So $\bigcup_n \mathcal{F}_n \subseteq G \subseteq \mathcal{F}$. Hence ϕ and Ω are in G. If $E \in G$ then

$$I_{\Omega \setminus E} = I_\Omega - I_E \in \{P(f) : f \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$$

because I_Ω and $I_E \in \{P(f) : f \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$ and this is a subspace. Finally note that if $(E_n) \subset G$, with $E_n \cap E_m = \phi$ for $m \neq n$ then

$$I_{\bigcup_{n=1}^{k} E_n} = \sum_{n=1}^{k} I_{E_n} \in \{P(f) : f \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$$

because ~~it~~ is a subspace, and, as $\mathbb{P}$ is a probability measure

$$\mathbb{P}\left(\bigcup_n E_n\right) = \sum_1^\infty \mathbb{P}(E_n)$$

$$\int (I_{\bigcup_n E_n})\, d\mathbb{P} = \lim_k \sum_1^k \int_\Omega I_{E_n}\, d\mathbb{P}$$

$$= \lim_k \int_\Omega \sum_{n=1}^{k} I_{E_n} \, d\mathbb{P} = \lim_k \int_\Omega I_{\bigcup_{n=1}^{k} E_n} \, d\mathbb{P}$$

or,

$$\lim_k \int_\Omega \left(I_{\bigcup_n E_n} - I_{\bigcup_{n=1}^{k} E_n} \right) d\mathbb{P} = 0$$

Now since indicator functions take only the values 0 and 1, and $\bigcup_n E_n \supseteq \bigcup_{n=1}^{k} E_n$, this is the same as

$$\lim_k \int_\Omega \left| I_{\bigcup_n E_n} - I_{\bigcup_{n=1}^{k} E_n} \right|^2 d\mathbb{P} = 0$$

So $I_{\bigcup_n E_n}$ is the limit in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of a sequence in $\{P(f) : f \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$. This last subspace is <u>closed</u> so $I_{\bigcup_n E_n}$ belongs to it and therefore $\bigcup_n E_n \in \mathcal{G}$. This shows that $\mathcal{G}$ is closed under countable disjoint unions. It follows that it is closed ~~an~~ under arbitrary countable unions (exercise). So $\mathcal{G}$ is a σ-field, $\bigcup_n \mathcal{F}_n \subseteq \mathcal{G} \subseteq \mathcal{F}$. But $\mathcal{F}$ is

As a consequence, $I_E \in \{P(f) : f \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$ for every $E \in \mathcal{F}$, so every <u>simple</u> random variable is in $\{P(f) : f \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$ and hence <u>every</u> random variable in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ must lie in this subspace for any f in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a limit in L^2 of simple random variables (exercise).

<u>Step 3</u> Clearly $\bigcup_n L^2(\Omega, \mathcal{F}_n, \mathbb{P}) \subseteq \{P(f) : f \in L^2(\Omega, \mathcal{F}, \mathbb{P})\} \subseteq \overline{\bigcup_n L^2(\Omega, \mathcal{F}_n, \mathbb{P})}^{\|\cdot\|}$

<u>Remark</u> This amounts to saying that the "projection" P is the identity, $f \mapsto f$.

so that ~~Finally~~ ~~then~~

$$\overline{\bigcup_n L^2(\Omega, \mathcal{F}_n, \mathbb{P})}^{\|\cdot\|} = L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

~~and~~ So, reverting to our previous notation, if $f \in L^2$ and $Z = \lim_n M_{\mathcal{F}_n}(f)$ then for any $h \in L^2$

$$\langle f - Z, h \rangle = \lim_n \langle f - M_{\mathcal{F}_n}(f), h \rangle$$

and there is a sequence in $\bigcup_n L^2(\Omega, \mathcal{F}_n, \mathbb{P})$, (h_k) say, with $h_k \to h$ and we can assume w.l.o.g. that $h_k \in L^2(\Omega, \mathcal{F}_{n_k}, \mathbb{P})$ and $n_k \to \infty$ as $k \to \infty$. Now for k large enough

$$\cancel{\langle M_{\mathcal{F}_n}(f), h_k \rangle = \langle M_{\mathcal{F}_n}(f), M_{\mathcal{F}_n}(h_k) \rangle}$$

$$\cancel{= \langle f, M_{\mathcal{F}_n}(h_k) \rangle}$$

$$\cancel{\to \langle f, M_{\mathcal{F}_n}(h) \rangle \text{ as } k \to \infty}$$

$\|h_k - h\| < \epsilon/2$, fix such a k.

Then choose $n > n_k$ so that, by $**$,

$$\langle M_{\mathcal{F}_n}(f), h_k \rangle = \langle f, h_k \rangle$$

Then

$$|\langle f-Z, h\rangle| \leq |\langle f-Z, h-h_k\rangle| + |\langle f-Z, h_k\rangle|$$

$$\leq \|f-Z\| \cdot \|h-h_k\| + |\langle f-Z, h_k\rangle|$$

Note $\langle f-Z, h_k\rangle = \lim_n \langle f - M_{\mathcal{F}_n}(f), h_k\rangle$

$$= \lim_n \left(\langle f, h_k\rangle - \langle M_{\mathcal{F}_n}(f), h_k\rangle\right)$$

$= 0$ since $\langle M_{\mathcal{F}_n}(f), h_k\rangle = \langle f, h_k\rangle$ for all large n.

So $|\langle f-Z, h\rangle| \leq \|f-Z\| \cdot \dfrac{\epsilon}{2}$ for each $\epsilon > 0$.

ie. $\langle f-Z, h\rangle = 0$ for every $h \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Consequently $f = Z$.

<u>Definition</u>

Let $(\mathcal{F}_n)$ and $\mathcal{F}$ be as before. A sequence (X_n) with $X_n \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ and $M_{\mathcal{F}_n}(X_{n+1}) = X_n$ is called a <u>martingale</u>.

<u>Proposition</u> (Martingale Convergence Theorem)

Let $X \in L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$. Then if $X_n = (M_{\mathcal{F}_n}(X))$ (X_n) is a martingale and $X_n \to X$ in L^2 mean.

<u>Proof</u> Lemma M2 and Theorem M2.

An application: Approximation of f for a a Riemann integrable function on $(0,1]$. Take $\mathcal{F}$ = Lebesgue sets of $(0,1]$ and $\mathcal{F}_n$ usual dyadic σ-field.

Let E be an interval, define the <u>average of f over E</u> to be $\frac{1}{\mathbb{P}(E)}\int_E f(x)dx$. Let $\mathcal{F}_n$ denote the σ-field given by the dyadic partition $\{(i-1/2^n, i/2^n] : 1 \leq i \leq 2^n\}$.

Let $f_n \equiv 2^n \int_{i-1/2^n}^{i/2^n} f(x)dx$ on $\left(\frac{i-1}{2^n}, \frac{i}{2^n}\right]$, $1 \leq i \leq 2^n$.

Riemann Integral $\equiv$ Lebesgue Integral.

Claim $f_n \to f$ in L^2 mean: Observe that $f_n \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ and if $g(x) = \sum_{i=1}^{2^n} g(i) I_{(i-1/2^n, i/2^n]}(x) \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$

then $\int f \, g \, d\mathbb{P} = \sum^{2^n} \left(2^n \int^{i/2^n} f(x)dx\right) g(i)$

$$= \sum_{i=1}^{2^n} \int_{i-1/2^n}^{i/2^n} g(i) f(x)\,dx = \int_0^1 g(x) f(x)\,dx = \langle f, g \rangle .$$

So $f_n = M_{\mathcal{F}_n}(f)$ by the relation $**$!.

It follows that $f_n \longrightarrow f$ in L^2-mean and hence in probability, by Martingale Convergence and 4.1.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\cancel{\mathcal{F} = \bigcup_n \mathcal{F}_n}$.

Let $X_1, X_2, \ldots, X_n,$ be a sequence of independent random variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ with $E(X_i) = 0$, $i \geq 1$. Each random variable, Y, determines a σ-field on Ω, namely that given by $\{ Y^{-1}(B) : B$ is a Borel subset of $\mathbb{R}\}$. Denote by $\mathcal{F}_n$, $\sigma(X_1, X_2, \ldots, X_n)$ — the σ-field generated by the sets $\{ X_i^{-1}(E) : 1 \leq i \leq n,\ E \subseteq \mathbb{R}$ a Borel set$\}$. Then $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for $n = 1, 2, \ldots$. Let $Y_n = \sum_{i=1}^{n} X_i$. Then (Y_n) is a martingale (with respect to the σ-fields $(\mathcal{F}_n)$), because,

$$M_{\mathcal{F}_n}(Y_{n+1}) = M_{\mathcal{F}_n}(X_1 + X_2 + \cdots + X_{n+1})$$
$$= M_{\mathcal{F}_n}(X_1 + X_2 + \cdots + X_n) + M_{\mathcal{F}_n}(X_{n+1})$$

Note that $X_1 \in L^2(\Omega, \mathcal{F}_1, \mathbb{P})$ because $\mathcal{F}_1$ contains all the sets $X_1^{-1}(s,t]$ $s<t$, $s,t \in \mathbb{R}$.

Similarly $X_1 + X_2 + \cdots + X_n \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$.

Now what is $M_{\mathcal{F}_n}(X_{n+1})$?, recall X_{n+1} is independent of all events in $\mathcal{F}_n$. So for s,t as above

$$\mathbb{P}\left(X_{n+1}^{-1}(s,t] \cap E \right) = \mathbb{P}\left(X_{n+1}^{-1}(s,t]\right) \cdot \mathbb{P}(E) \quad \forall E \in \mathcal{F}_n$$

Put another way,

$$\int_\Omega I_{\{X_{n+1}^{-1}(s,t]\}} I_E \, d\mathbb{P} = \left(\int_\Omega I_{\{X_{n+1}^{-1}(s,t]} \, d\mathbb{P}\right)\left(\int_\Omega I_E \, d\mathbb{P}\right)$$

Now we write out this for a simple r.v. which will be an approximation to X_{n+1}, namely

$$\sum_{i=-k2^k}^{i=k2^k-1} i/2^k \, I_{\{X_{n+1}^{-1}(i/2^k, i+1/2^k]}.$$

Then

$$\int_\Omega \sum_{i=-k2^k}^{k2^k-1} i/2^k \, I_{\{X_{n+1}^{-1}(i/2^k, i+1/2^k]\}} \cdot I_E \, d\mathbb{P} =$$

$$= \sum^{k2^k-1} \int i/2^k \, I_{\{\cdots\}} \cdot I_E \, d\mathbb{P} =$$

$$\sum_{i=-k2^k}^{k2^k-1} i/2^k \, \mathbb{P}\{X_{n+1}^{-1}(i/2^k, i+1/2^k]\} \, \mathbb{P}(E) = \int_{\Omega} \sum_i i/2^k I_{\{X_{n+1}^{-1}(i/2^k, i+1/2^k]\}} \, d\mathbb{P} \times \mathbb{P}(E).$$

Taking the limit on k gives

$$\langle X_{n+1}, I_E \rangle = \left(\int_{\Omega} X_{n+1} \, d\mathbb{P} \right) \mathbb{P}(E)$$

$$= 0. \quad \forall E \in \mathcal{F}_n.$$

So for any simple r.v., g, in $L^2(\Omega, \mathcal{F}_n, \mathbb{P})$

$\langle X_{n+1}, g \rangle = 0$ and by approximation

$\langle X_{n+1}, g \rangle = 0$ for all $g \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$.

So $\langle M_{\mathcal{F}_n}(X_{n+1}), g \rangle = \langle X_{n+1}, g \rangle = 0$ for $g \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$, i.e. $M_{\mathcal{F}_n}(X_{n+1}) = 0$.

So (Y_n) is a martingale.

We can now deduce that if $\sup_n \|Y_n\| < \infty$ then (Y_n) converges to some random variable, Y, in $L^2(\Omega, \sigma(\bigcup_n \mathcal{F}_n), \mathbb{P})$. To see this look at the Proof of the first part of M2, the sequence of numbers, $(\|Y_n\|^2)$, is increasing and by hypothesis bounded above. As in the proof of M2,

$$\|Y_n - Y_m\|^2 = \|Y_n\|^2 - \|Y_m\|^2$$

So (Y_n) is L^2 - Cauchy. (We note that it is not necessary to use the martingale convergence theorem to prove this result, but that is beside the point).