So, $M_G(f) I_{\{M_G(f)<0\}}$ is a function taking negative values only and whose integral is zero.

It follows that $M_G(f) I_{\{M_G(f)<0\}} \equiv 0$ (is equivalent to 0) so $M_G(f) \geq 0$.

(iv) If $f(\omega) \leq C \ \forall \omega \in \Omega$ then $C I_\Omega - f \geq 0$

thus $M_G(C I_\Omega - f) \geq 0$ by (iii), using (i) we get $C M_G(I_\Omega) - M_G(f) \geq 0$ and using (v)

$$C - M_G(f) \geq 0$$

or

$$C \geq M_G(f) .$$

(vii) We show that $M_G(fg) - M_G(f) \cdot g$ is zero.

Let $k \in L^2(\Omega, G, \mathbb{P})$, then $gk \in L^2(\Omega, G, \mathbb{P})$ and

$$\langle M_G(fg) - M_G(f) \cdot g, k \rangle = \langle M_G(fg), k \rangle - \langle M_G(f) g, k \rangle$$

$= \langle fg, k \rangle - \langle M_G(f), gk \rangle$ by ** & property of $\int_\Omega \cdot \, d\mathbb{P}$

$= \langle fg, k \rangle - \langle f, gk \rangle$ by **

$= \langle fg, k \rangle - \langle fg, k \rangle$ $\left(\langle fg, k \rangle = \int_\Omega (fg) k \, d\mathbb{P} = \int_\Omega f(gk) \, d\mathbb{P} \right)$

$= 0$.

(viii) $\mathbb{E}(M_G(f)) = \int_\Omega M_G(f) \, d\mathbb{P} = \int_\Omega M_G(f) I_\Omega \, d\mathbb{P} \overset{\uparrow}{=} \langle M_G(f), I_\Omega \rangle$

$= \langle f, M_G(I_\Omega) \rangle$

Now we can return to our family of σ-fields, $\mathcal{F}_n$, $n = 1, 2, 3, \ldots$ For any f in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ our "estimates" of f are given by the sequence $M_{\mathcal{F}_n}(f)$. Now the distance between f and $M_{\mathcal{F}_n}(f)$ gets smaller as n gets larger — this should be clear since $M_{\mathcal{F}_n}(f)$ is the element of $L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ closest to f and $L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ is a subspace of $L^2(\Omega, \mathcal{F}_{n+1}, \mathbb{P})$, so $M_{\mathcal{F}_{n+1}}(f)$ should be closer to f than $M_{\mathcal{F}_n}(f)$. It seems reasonable to suppose then that if $\lim_n M_{\mathcal{F}_n}(f)$ exists it will be our "best" estimate of f given all the information in $\bigcup_n \mathcal{F}_n$. Before we take this further let us look at some explicit examples of conditional expectations.

Example

Let us consider the σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\Omega = (0, 1]$ discussed in our last example. Recall that $\mathcal{F}_1 = \{\phi, \Omega, (0, \frac{1}{2}], (\frac{1}{2}, 1]\}$ while $\mathcal{F}_2 = \{\phi, \Omega, (0, \frac{1}{4}], (\frac{1}{4}, \frac{1}{2}], (\frac{1}{2}, \frac{3}{4}], (\frac{3}{4}, 1]\}$. Let M denote the conditional expectation of $L^2(\Omega, \mathcal{F}_2, \mathbb{P})$ onto $L^2(\Omega, \mathcal{F}_1, \mathbb{P})$, of course $\mathbb{P}$ is Lebesgue measure. Now the random variables in $L^2(\Omega, \mathcal{F}_2, \mathbb{P})$ have a simple form, if f is a typical element of $L^2(\Omega, \mathcal{F}_2, \mathbb{P})$ then, while any $g \in L^2(\Omega, \mathcal{F}_1, \mathbb{P})$ has the form,

$$g(\omega) = \sum_{i=1}^{2} \beta_i I^{(\omega)}_{(i-1/2,\, i/2]} .$$

So $M(f)$ will have this form too.

Qu. Can we determine $M(f)$?. The answer is yes and we use the relationship $**$,

$$\langle M(f), g \rangle = \langle f, g \rangle \quad \begin{cases} f \in L^2(\Omega, \mathcal{F}_2, \mathbb{P}) \\ g \in L^2(\Omega, \mathcal{F}_1, \mathbb{P}) \end{cases}$$

which maybe rewritten,

$$\int_\Omega M(f) g \, d\mathbb{P} = \int_\Omega f g \, d\mathbb{P} .$$

If $f = \sum_{i=1}^{4} \alpha_i I_{(i-1/4,\, i/4]}$ and $g = I_{(0,1/2]}$ this yeilds,

$$\int_\Omega M(f) I_{(0,1/2]} \, d\mathbb{P} = \frac{\alpha_1}{4} + \frac{\alpha_2}{4} .$$ Now if

$M(f) = \beta_1 I_{(0,1/2]} + \beta_2 I_{(1/2,1]}$ then $\int_\Omega M(f) I_{(0,1/2]} \, d\mathbb{P} = \frac{\beta_1}{2}$

So $\beta_1 = \frac{\alpha_1 + \alpha_2}{2}$, and, similarly, $\beta_2 = \frac{\alpha_3 + \alpha_4}{2}$.

Thus if $f = \sum_{i=1}^{4} \alpha_i I_{(i-1/2^2,\, i/2^2]}$ then $M(f) = \left(\frac{\alpha_1 + \alpha_2}{2}\right) I_{(0,1/2]} +$ $\left(\frac{\alpha_3 + \alpha_4}{2}\right) I_{(1/2,1]}$. So the expectation is obtained by <u>averaging</u> the values of f_2 over the sets in $\mathcal{F}_1$. Could this result be of a more general character ? :

Any f in $L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ has the form $\sum_{i=1}^{2^n} \alpha_i I_{(i-1/2^n, i/2^n]}$, if M is the conditional expectation of $L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ onto $L^2(\Omega, \mathcal{F}_m, \mathbb{P})$, where $m < n$, then knowing that $M(f) = \sum_{i=1}^{2^m} \beta_i I_{(i-1/2^m, i/2^m]}$ we know that,

$$\int_\Omega M(f) I_{(i-1/2^m, i/2^m]} d\mathbb{P} = \int_\Omega f I_{(i-1/2^m, i/2^m]} d\mathbb{P}$$

ie.

$$\beta_i/2^m = \frac{\alpha_{(i-1)2^{n-m}}}{2^n} + \frac{\alpha_{(i-1)2^{n-m}+1}}{2^n} + \cdots \cdots + \frac{\alpha_{i2^{n-m}}}{2^n} \quad (2^{n-m} \text{ terms})$$

So that

$$\beta_i = \frac{\alpha_{(i-1)2^{n-m}} + \alpha_{(i-1)2^{n-m}+1} + \cdots + \alpha_{i2^{n-m}}}{2^{n-m}}$$

ie. β_i is the average of the values of f on $I_{(i-1/2^n, i/2^n]}$. This gives an alternative way of viewing conditional expectations, they arise from <u>averaging</u> <u>processes</u>.

Let us return now to the situation $(\mathcal{F}_n)$ is an 'increasing sequence of σ-fields on some set, Ω, and $\mathbb{P}$ is a probability measure on a σ-field $\mathcal{F} \supseteq \bigcup_n \mathcal{F}_n$. We will assume, from this point on, that $\mathcal{F}$ is the σ-field generated by $\bigcup_n \mathcal{F}_n$. One would hope that with this assumption we could get good approximation /estimates of f. [Certainly, if $\mathcal{F}$ is very much larger than the σ-field generated by $\bigcup_n \mathcal{F}_n$ we cannot expect that the sequence of conditional expectations will give a good approximation to $f \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, you can see this by considering the special case where $\mathcal{F}_n$ is the "dyadic" σ-field on $(0,1]$, for $m < n$ and $n-m$ "large enough" the average of the values of $f \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ on the sets in $\mathcal{F}_m$ can be arranged to be quite large (try this as an exercise).].

Theorem (M2)

Let $f \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Then

$$\| M_{\mathcal{F}_n}(f) - f \|_2 \longrightarrow 0 \quad \text{as } n \to \infty .$$

$\boxed{\text{INSERT}}$

Proof

Note first of all that $\forall n \quad \| M_{\mathcal{F}_n}(f) \|^2 \leq \|f\|^2$ (M1(ii)). Secondly, observe that, if $m < n$, then our lemma tells us that,

$$\| M_{\mathcal{F}_m}(f) \|^2 = \langle M_{\mathcal{F}_m}(f), M_{\mathcal{F}_m}(f) \rangle = \langle M_{\mathcal{F}_n} \circ M_{\mathcal{F}_m}(f), M_{\mathcal{F}_n} \circ M_{\mathcal{F}_m}(f) \rangle$$

INSERT

Lemma (M2).

If $m \leqslant n$ then $M_{\mathcal{F}_m} \circ M_{\mathcal{F}_n} = M_{\mathcal{F}_n} \circ M_{\mathcal{F}_m} = M_{\mathcal{F}_m}$.

Proof

Let $k \in L^2(\Omega, \mathcal{F}_m, \mathbb{P})$. Since $\mathcal{F}_m \subseteq \mathcal{F}_n$ then

$M_{\mathcal{F}_n}(M_{\mathcal{F}_m}(f)) = M_{\mathcal{F}_m}(f)$ because $L^2(\Omega, \mathcal{F}_m, \mathbb{P}) \subseteq L^2(\Omega, \mathcal{F}_n, \mathbb{P})$ and M1(vi) holds, in short, $M_{\mathcal{F}_n}(h) = h$ for $h \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$.

So $M_{\mathcal{F}_n} \circ M_{\mathcal{F}_m} = M_{\mathcal{F}_m}$. Now, using $**$, (recall $**$ is $\langle P_K x, y \rangle = \langle x, y \rangle \; \forall y \in K$).

$$\langle M_{\mathcal{F}_m} \circ M_{\mathcal{F}_n}(f), k \rangle = \langle M_{\mathcal{F}_n}(f), k \rangle \quad \text{by } **$$

$$= \langle f, k \rangle \quad \text{again by } **$$

$$= \langle M_{\mathcal{F}_m}(f), k \rangle \quad \text{again by } ** .$$

So $M_{\mathcal{F}_m} \circ M_{\mathcal{F}_n} = M_{\mathcal{F}_m}$.

So $(\| M_{\mathcal{F}_n}(f) \|^2)$ is an increasing sequence of real numbers.

Now $\| M_{\mathcal{F}_n}(f) - M_{\mathcal{F}_m}(f) \|^2 = \langle M_{\mathcal{F}_n}(f) - M_{\mathcal{F}_m}(f), M_{\mathcal{F}_n}(f) - M_{\mathcal{F}_m}(f) \rangle$

expanding we get,

$$= \langle M_{\mathcal{F}_n}(f), M_{\mathcal{F}_n}(f) \rangle - \langle M_{\mathcal{F}_n}(f), M_{\mathcal{F}_m}(f) \rangle - \langle M_{\mathcal{F}_m}(f), M_{\mathcal{F}_n}(f) \rangle + \langle M_{\mathcal{F}_m}(f), M_{\mathcal{F}_m}(f) \rangle$$

Now $\langle M_{\mathcal{F}_n}(f), M_{\mathcal{F}_m}(f) \rangle = \langle M_{\mathcal{F}_n}(f), M_{\mathcal{F}_n}(M_{\mathcal{F}_m}(f)) \rangle$

$$= \langle f, M_{\mathcal{F}_m}(f) \rangle \quad \text{by } **$$

$$= \langle M_{\mathcal{F}_m}(f), M_{\mathcal{F}_m}(f) \rangle \quad \text{by } **$$

So then $\langle M_{\mathcal{F}_m}(f), M_{\mathcal{F}_n}(f) \rangle = \langle M_{\mathcal{F}_m}(f), M_{\mathcal{F}_m}(f) \rangle$. Hence,

$$\| M_{\mathcal{F}_n}(f) - M_{\mathcal{F}_m}(f) \|^2 = \langle M_{\mathcal{F}_n}(f), M_{\mathcal{F}_n}(f) \rangle - \langle M_{\mathcal{F}_m}(f), M_{\mathcal{F}_m}(f) \rangle$$

$$= \| M_{\mathcal{F}_n}(f) \|^2 - \| M_{\mathcal{F}_m}(f) \|^2 .$$

Now $(\| M_{\mathcal{F}_n}(f) \|^2)$ is increasing and bounded above by $\| f \|^2$. So it is a Cauchy sequence. We conclude that $(M_{\mathcal{F}_n}(f))$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Let $Z = \lim_n M_{\mathcal{F}_n}(f)$. We show, finally that $Z = f$. To do this we will emplo-

You have met this in the following form.

For real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leqslant \left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |b_i|^2\right)^{1/2}$$

This result generalises to

$$|\langle f, g \rangle| \leqslant \|f\| \, \|g\|$$

or

$$\left|\int_\Omega fg \, d\mathbb{P}\right| \leqslant \left(\int_\Omega |f|^2 d\mathbb{P}\right)^{1/2} \left(\int_\Omega |g|^2 d\mathbb{P}\right)^{1/2}$$

For $f, g \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. We require this to make one point only. Suppose $f_n \to f$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Then
$$|\langle f, g\rangle - \langle f_n, g\rangle| = |\langle f - f_n, g\rangle|$$
$$\leqslant \|f - f_n\| \cdot \|g\|$$
$$\to 0 \text{ as } n \to \infty.$$

So, $f_n \to f$ then $\langle f_n, g\rangle \to \langle f, g\rangle$ $\forall g \in L^2(\Omega, \mathcal{F}, \mathbb{P})$

Hence $\langle M_{\mathcal{F}_n}(f), g\rangle \longrightarrow \langle Z, g\rangle$ for each $g \in L^2(\Omega, \mathcal{F}, \mathbb{P})$

In particular, if $g \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$, for any $n \in \mathbb{N}$